Tutorial 8: Planarity, Kuratowski's Theorem, Hamilton Cycles and Trees

Euler's formula and the edge bounds, subdivisions and Kuratowski, complements, Hamilton paths and cycles, pendant vertices in trees, orientations, and Rédei's theorem

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Draw a **planar embedding** of a solid such as the octahedron, and count its vertices, edges and **regions** with Euler's formula $v - e + r = 2$.
2. Decide whether a graph *could* be planar from its degree sequence, and know exactly **what the edge bound does and does not prove**.
3. Prove that every simple planar graph has a vertex of degree at most 5, using the average degree.
4. Use **elementary subdivisions** and **Kuratowski's theorem** to prove that the **Petersen graph** is not planar, by exhibiting a subgraph homeomorphic to $K_{3,3}$.
5. Compute the **complement** of $K_{m,n}$ and read off its longest path and longest cycle.
6. Prove that the hypercube Q_n is **bipartite** and use parity as a proof tool.
7. Repair a graph so that it has an **Eulerian trail** (the Königsberg bridge-builder), and say exactly how many edges you must add.
8. Translate a seating puzzle into a **Hamilton cycle** question, and prove that a Hamilton cycle does *not* exist using a cut vertex, a degree-1 vertex or a bipartite imbalance.
9. Decide for which n the graph K_n has a Hamilton path and a Hamilton cycle.
10. Prove **Rédei's theorem**: every tournament has a directed Hamilton path.
11. Prove the five **additional** results: pendant vertices in a tree, a triangle-free bipartiteness criterion, balanced orientations, the $n^2/4$ bound for bipartite graphs, and embedding a graph into a regular graph.

Sections 1.1 to 1.7 review the tools, each with the intuition, a simple example, the formal statement, a comprehensive example and a recap. Section 2 solves Problems P1 to P10 of the *Graphs Tutorial 3* sheet in full. Section 3 solves the five problems of the *Additional* sheet. Section 4 collects the classic mistakes, "spot the error" puzzles, a checklist and exercises. Appendix A answers the puzzles and Appendix B answers the exercises, so the whole document can be used for **self-study**. Tutorial 7 (degrees, Handshaking, walks and trails, Euler's theorem, bipartite and planar graphs) is assumed; everything it introduced is restated here before it is used.

Conventions used throughout

Unless a problem says otherwise, a **graph** $G = (V, E)$ is finite, undirected and **simple**: no loops and no multiple edges. We write $v = |V|$, $e = |E|$, and $u \sim w$ when $\{u, w\} \in E$. We write $N(x)$ for the set of neighbours of x and $\deg(x) = |N(x)|$; $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree.

The **length** of a path or cycle is its **number of edges**, so a path of length k has $k + 1$ vertices. Following the lectures, the regions of a planar embedding are written r (many books call them *faces* and write f); the two words mean the same thing. Königsberg is a **multigraph**, and that is said explicitly where it matters.

Theorem numbers such as *Theorem 11.6* and *Corollary 11.3* refer to the numbering used in the lectures (Chapters 11 to 13 of the course text).

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1 The toolbox

Tutorial 7 ended with two families of graphs, bipartite and planar, and with Euler's formula. This tutorial pushes both much further. Sections 1.1 to 1.3 are the planarity toolkit: how to *count* in a planar drawing, how to use the count to prove that something is **not** planar, and what to do when counting is not enough (Kuratowski). Section 1.4 is a short detour on complements, Section 1.5 is Hamilton's question ("every vertex once"), Section 1.6 collects the facts about trees that the additional problems need, and Section 1.7 moves to directed graphs. The examples in this section are *not* the tutorial problems; those are solved in Sections 2 and 3.

1.1 Planar embeddings, regions, and Euler's formula

The intuition

A graph is **planar** if it *can* be drawn on a flat sheet with edges meeting only at shared endpoints. Such a drawing is an **embedding**. Once you have an embedding, the edges cut the sheet into **regions**, and you must remember to count the **unbounded outside region** as one of them.
Euler's observation is that the number of regions is not up to you. Vertices, edges and regions are locked together by $v - e + r = 2$. Build the drawing one edge at a time: every new edge either reaches a brand new vertex (so v and e both go up by 1, and $v - e$ is unchanged) or closes a loop back into the picture (so it cuts an existing region in two: e and r both go up by 1, and $r - e$ is unchanged). Either way $v - e + r$ never moves, and at the very start, with one vertex and no edges, it equals $1 - 0 + 1 = 2$.

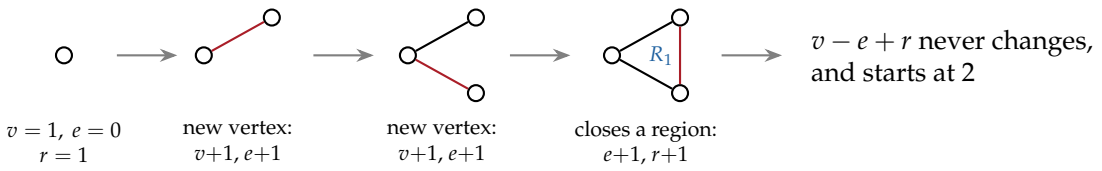


Figure 1. Why $v - e + r = 2$. Each new edge (red) either brings a new vertex or closes a new region. The outside region is always counted.

Simple example: a triangle with a tail

The last picture of Figure 1 has $v = 3$, $e = 3$ and $r = 2$ (inside the triangle, and outside): $3 - 3 + 2 = 2$. ✓ Attach a pendant edge to one corner: now $v = 4$, $e = 4$, and r is *still* 2, because a pendant edge cuts no region in two. Indeed $4 - 4 + 2 = 2$. ✓

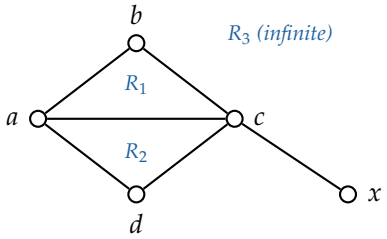
Planar graphs, regions, Euler's formula

A graph or multigraph G is **planar** if it can be drawn in the plane with its edges intersecting only at vertices of G; such a drawing is a **planar embedding** of G. An embedding divides the plane into **regions**, one of which is unbounded (the **infinite region**).
Theorem 11.6 (Euler). If G is a **connected** planar graph or multigraph with v vertices and e edges, then every planar embedding of G has exactly

$$r = e - v + 2 \qquad \text{regions, that is} \qquad v - e + r = 2.$$

The **degree of a region** R, written $\deg(R)$, is the number of edges met when you walk once around its boundary; an edge with the same region on both sides is counted **twice**. Since every edge borders exactly two region-sides,

$$\sum_{\text{regions } R} \deg(R) = 2e.$$



Region	deg	boundary walk
R_1	3	a, b, c, a
R_2	3	a, c, d, a
R_3	6	a, b, c, x, c, d, a
total	12	$= 2e = 2 \cdot 6$ ✓

*Figure 2. Degrees of regions. The pendant edge cx is walked **twice** by the infinite region, once on each side, which is why R_3 has degree 6 and not 5. Check: $v - e + r = 5 - 6 + 3 = 2$.*

Comprehensive example: how many regions does the cube have?

The cube graph Q_3 has $v = 8$ and $e = 12$, and it is connected and planar (Tutorial 7, Figure 7: a square inside a square). Euler's formula gives $r = e - v + 2 = 12 - 8 + 2 = 6$, which matches the six faces of the solid cube: the inner square, the four trapezoids between the squares, and the outside. The sum of the region degrees is $6 \cdot 4 = 24 = 2 \cdot 12$. ✓

What the formula is really saying. For a *solid*, the regions of the planar drawing are exactly the faces of the solid, with the face you "look through" stretched out into the infinite region. That is the trick for Problem P1.

Three traps

(i) Count the outside. Forgetting the infinite region turns $v - e + r = 2$ into $v - e + r = 1$.

(ii) Connected. Euler's formula needs G connected. For a planar graph with k connected components the formula becomes $v - e + r = 1 + k$.

(iii) A drawing with crossings proves nothing. "Planar" means a crossing-free drawing *exists*.

Euler's formula in one breath

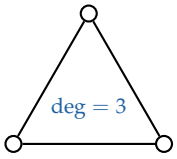
- Connected planar: $v - e + r = 2$, counting the infinite region. k components: $v - e + r = 1 + k$.
- $\sum_R \deg(R) = 2e$, where an edge on the boundary of only one region is counted twice.

1.2 The edge bounds, and the vertex of degree at most five

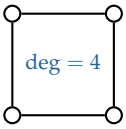
The intuition

A planar graph cannot be crowded. Every region needs a boundary, and in a simple graph the shortest possible boundary is a triangle, so each region "uses up" at least 3 edge-sides. There are only $2e$ edge-sides to go around, so $3r \leq 2e$. Feeding that into $v - e + r = 2$ gives a ceiling on the number of edges: $e \leq 3v - 6$.

If the graph has **no triangles** the shortest boundary is a quadrilateral, so each region uses at least 4 edge-sides, $4r \leq 2e$, and the ceiling drops to $e \leq 2v - 4$. Fewer edges allowed means the bound is *easier to violate*, which is exactly what you want when you are trying to prove non-planarity. A ceiling on edges is also a ceiling on the *average* degree, and the minimum is never above the average. That is the whole proof that a planar graph always has a vertex of degree at most 5.



smallest region
in a simple graph



smallest region if
there is no triangle

$$3r \leq \sum_R \deg(R) = 2e, \text{ so } r \leq \tfrac{2e}{3}$$

$$2 = v - e + r \leq v - e + \tfrac{2e}{3} = v - \tfrac{e}{3}$$

$$\Rightarrow \boxed{e \leq 3v - 6} \qquad \text{(no triangle: } \boxed{e \leq 2v - 4}\text{)}$$

Figure 3. Where the edge bounds come from. Both need G simple and connected with $v \geq 3$.

Simple example: K_5 fails the bound, $K_{3,3}$ needs the better one

K_5: $v = 5$, $e = 10$, and $3v - 6 = 9 < 10$. Not planar. ✓
$K_{3,3}$: $v = 6$, $e = 9$, and $3v - 6 = 12 \geq 9$, so this bound says *nothing*. But $K_{3,3}$ is bipartite, so it has no triangle, and $2v - 4 = 8 < 9$. Not planar. ✓

Edge bounds and the minimum degree

Let G be a **simple** planar graph (no loops, no multiple edges) with $v \geq 3$ vertices and e edges.

Corollary 11.3.	$3r \leq 2e$	and	$e \leq 3v - 6$.	
Triangle-free version.	$4r \leq 2e$	and	$e \leq 2v - 4$	(for instance when G is bipartite).
Girth k version.	$e \leq \frac{k}{k-2}(v-2)$		when G has a cycle and every cycle has length $\geq k$.	

Minimum degree. Every simple planar graph satisfies $\delta(G) \leq 5$.
The bounds are stated for connected G in the lectures, but they hold for *every* simple planar graph with $v \geq 3$: adding edges between different components keeps the graph planar and only increases e, so the bound for the connected graph you end up with also bounds the one you started from.

Comprehensive example: the bounds are tight

(a) $e = 3v - 6$ is attained. The octahedron (Problem P1) has $v = 6$ and $e = 12 = 3 \cdot 6 - 6$. Every region of its embedding is a triangle; such a graph is called a **triangulation**, and triangulations are exactly the planar graphs with the maximum possible number of edges.
(b) $e = 2v - 4$ is attained. The cube Q_3 has $v = 8$ and $e = 12 = 2 \cdot 8 - 4$, and it is bipartite, so triangle-free.
(c) $\delta \leq 5$ is tight. The icosahedron is planar with $v = 12$, $e = 30$, $r = 20$, and it is 5-regular, so no planar graph can be guaranteed a vertex of degree 4 or less.

The direction of the implication

$$G \text{ planar} \implies e \leq 3v - 6, \qquad \text{but } \textbf{not} \text{ the converse.}$$

So a bound that **fails** proves non-planarity, and a bound that **holds** proves *nothing at all*. $K_{3,3}$ satisfies $e \leq 3v - 6$ and is not planar; the graph of Problem P2 satisfies it and *is* planar. When the bound holds you must either draw the graph (to prove planar) or find a Kuratowski subgraph (to prove non-planar).

Edge bounds in one breath

- Simple planar, $v \geq 3$: $e \leq 3v - 6$; triangle-free as well: $e \leq 2v - 4$; girth $\geq k$ *and at least one cycle*: $e \leq \frac{k}{k-2}(v-2)$. (Without a cycle it fails: the path on 3 vertices has $e = 2 > \frac{5}{3}$.)
- Average degree $= \frac{2e}{v} \leq 6 - \frac{12}{v} < 6$, hence $\delta(G) \leq 5$.
- A violated bound disproves planarity. A satisfied bound decides nothing.

1.3 Subdivisions, homeomorphic graphs, and Kuratowski's theorem

The intuition

Putting an extra dot in the middle of an edge changes nothing about whether a graph can be drawn without crossings: you have just marked a point that was already on the curve. That operation is an **elementary subdivision**, and two graphs that are the same "up to extra dots on edges" are called **homeomorphic**.
Kuratowski's theorem says that this is the *whole* story of non-planarity. There are exactly two obstructions, K_5 and $K_{3,3}$, and a graph is non-planar precisely when one of them is hiding inside it, possibly with extra dots on its edges. To use it you pick 6 (for $K_{3,3}$) or 5 (for K_5) **branch vertices** and find internally disjoint paths joining them in the right pattern.

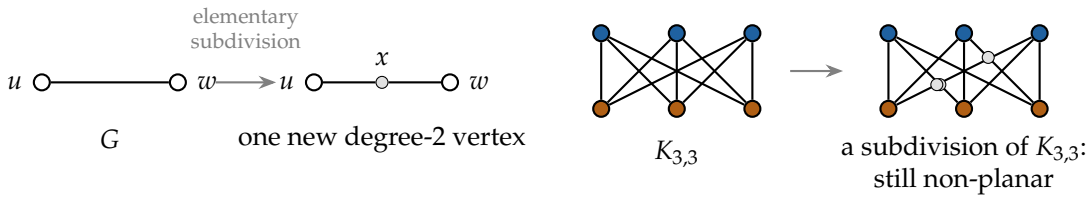


Figure 4. Left: one elementary subdivision. Right: $K_{3,3}$ and a graph homeomorphic to it. Grey dots are the degree-2 subdivision vertices; the coloured dots are the six ***branch vertices****.*

Simple example: reading a subdivision

A graph that is a subdivision of $K_{3,3}$ has exactly 6 vertices of degree 3 (the branch vertices) and all its other vertices have degree 2. So the first thing to do when you hunt for one is to **list the degree-3 vertices**: there must be exactly six, split $3 + 3$, with nine internally disjoint paths between the two sides.

Elementary subdivision, homeomorphic, Kuratowski

Let $G = (V, E)$ be a loop-free undirected graph with $E \neq \emptyset$. An **elementary subdivision** of G removes an edge $e = \{u, w\}$ and adds a **new** vertex $x \notin V$ together with the two edges $\{u, x\}$ and $\{x, w\}$.
Two loop-free undirected graphs G_1 and G_2 are **homeomorphic** if they are isomorphic, or if both can be obtained from the same loop-free undirected graph H by a sequence of elementary subdivisions. Informally, homeomorphic graphs are isomorphic "except possibly at vertices of degree 2". Homeomorphic graphs are either **both** planar or **both** non-planar.
Kuratowski's Theorem. A graph is **non-planar if and only if** it contains a subgraph homeomorphic to K_5 or to $K_{3,3}$.

Comprehensive example: a Kuratowski subgraph in K_6

K_6 contains K_5 as a subgraph (delete one vertex), so K_6 is non-planar: here the subgraph is literally K_5, with no subdivision needed. More generally K_n for $n \geq 5$ and $K_{m,n}$ for $m, n \geq 3$ are non-planar for the same reason.
How to search in practice. (1) Count: if $e > 3v - 6$, stop, the graph is non-planar and you do not need Kuratowski. (2) If the graph is triangle-free, try $e > 2v - 4$. (3) If the counts are inconclusive, try to draw it. (4) If drawing keeps failing, look for $K_{3,3}$: find six vertices of degree ≥ 3, split them $3 + 3$, and try to join every cross pair by paths that share no interior vertices. Deleting a few edges first often makes the pattern visible, and deleting edges is allowed because any subgraph of a planar graph is planar.

Two things Kuratowski does not say

(i) "Contains $K_{3,3}$" is not "contains $K_{3,3}$ as a subgraph". The subgraph only has to be *homeomorphic* to $K_{3,3}$, so its nine connections may be long paths rather than single edges. The Petersen graph of Problem P4 contains no $K_{3,3}$ at all as a plain subgraph, but it does contain a subdivision of one.
(ii) It is not a licence for circular reasoning. Writing "K_5 is non-planar by Kuratowski's theorem" is empty: the theorem is about containing K_5. Prove the two base cases with the edge bounds, as in Tutorial 7.

Kuratowski in one breath

- Elementary subdivision: replace an edge by a path of length 2 through a new vertex. Homeomorphic: same graph up to degree-2 vertices.
- Non-planar $\iff$ contains a subgraph homeomorphic to K_5 or $K_{3,3}$.
- To use it: name the branch vertices and the internally disjoint paths. To refute planarity faster, try the edge bound first.

1.4 Complements

The intuition

The **complement** $\overline{G}$ keeps the same vertices and swaps every edge for a non-edge. A useful way to picture it: take K_n on the same vertex set, and give G and $\overline{G}$ one half of its edges each. So $|E(G)| + |E(\overline{G})| = \binom{n}{2}$, and a question about "the pairs that are *not* joined" becomes a question about an ordinary graph.
Complementing turns "all edges go *between* the two parts" into "all edges stay *inside* the parts", which is why the complement of a complete bipartite graph falls apart into two complete graphs.

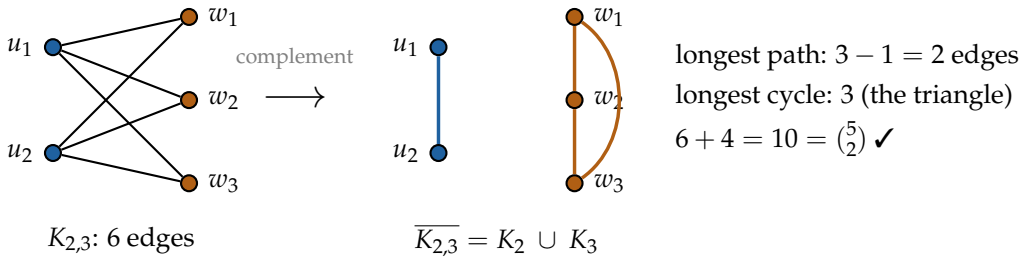


Figure 5. Complementing $K_{2,3}$. Every cross edge disappears and every inside non-edge appears, so the two parts become two complete graphs with nothing between them.

Simple example: $\overline{K_n}$ and $\overline{C_5}$

$\overline{K_n}$ is the edgeless graph on n vertices. And $\overline{C_5} \cong C_5$: a graph isomorphic to its own complement is called **self-complementary**, which forces $\binom{n}{2}$ to be even.

Complement

For a simple graph $G = (V, E)$ on n vertices, the **complement** is $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{\, \{x, y\} : x \neq y,\ \{x, y\} \notin E \,\}$. Hence

$$|E(G)| + |E(\overline{G})| = \binom{n}{2}, \qquad \deg_{\overline{G}}(x) = (n-1) - \deg_G(x) \quad \text{for every } x.$$

Writing $G_1 \cup G_2$ for the **disjoint union** (the two graphs side by side with no edge between them):

$$\overline{K_{m,n}} = K_m \cup K_n.$$

Comprehensive example: paths and cycles in a disjoint union

A path or a cycle is connected, so it lives **entirely inside one component**. In $K_m \cup K_n$ the components are K_m and K_n, and the longest path inside K_t runs through all t of its vertices, giving length $t-1$; the longest cycle inside K_t has length t when $t \geq 3$, and K_t has no cycle at all when $t \leq 2$. Therefore

$$\text{longest path in } K_m \cup K_n = \max(m, n) - 1, \qquad \text{longest cycle} = \begin{cases} \max(m, n), & \max(m, n) \geq 3, \\ \text{none}, & \max(m, n) \leq 2. \end{cases}$$

Moreover a complete graph K_t contains a path of *every* length $0, 1, \ldots, t-1$ and a cycle of *every* length $3, 4, \ldots, t$, so nothing between the extremes is missing. This single observation answers all of Problem P5.

Complements in one breath

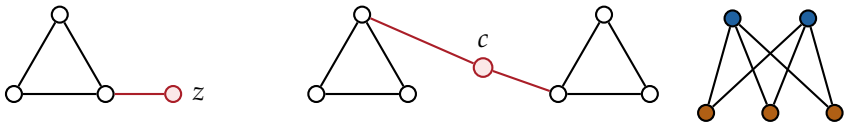
- $\deg_{\overline{G}}(x) = n - 1 - \deg_G(x)$ and $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.
- $\overline{K_{m,n}} = K_m \cup K_n$: two complete graphs with nothing in between.
- Paths and cycles cannot cross between components.

1.5 Hamilton paths and Hamilton cycles

The intuition

Euler's question was about **edges**: use every edge exactly once. Hamilton's question is about **vertices**: visit every vertex exactly once. The two look like twins, and they behave completely differently. Euler's question has a clean answer you can check in one pass over the degrees. Hamilton's question has no such test, and deciding it is one of the standard hard problems of computer science.

So in practice you work from both ends. To prove "yes", **exhibit the cycle** and check it. To prove "no", find an **obstruction**: a vertex of degree 1, a cut vertex, or an imbalance between the two sides of a bipartite graph. Each obstruction is a short, complete argument, and each one is a sentence you can reuse.



(i) $\deg(z) = 1$ (ii) c is a cut vertex (iii) $|X| = 2 \neq 3 = |Y|$

Figure 6. Three standard reasons a Hamilton cycle cannot exist. (i) A cycle needs 2 edges at every vertex. (ii) c is one of its cut vertices: deleting c leaves 2 pieces, but a cycle survives the deletion of one vertex in one piece. (iii) A cycle in a bipartite graph alternates sides, so the sides must have equal size.

Simple example: Euler yes, Hamilton no, and the other way round

$K_{1,3}$ (the star with three leaves) has three odd vertices, so no Eulerian trail, and it has a degree-1 vertex so no Hamilton cycle. The bowtie of Tutorial 7 has an Eulerian tour but no Hamilton cycle (its centre would need four edges on the cycle). The cube Q_3 has a Hamilton cycle but no Eulerian trail (all eight degrees are odd).

Hamilton paths and cycles, and the standard tests

A **Hamilton path** in G is a path that contains every vertex of G exactly once; a **Hamilton cycle** is a cycle that contains every vertex exactly once. A graph with a Hamilton cycle is called **Hamiltonian**.

Necessary conditions (how to prove "no"). If G has a Hamilton cycle then

(i) G is connected and $\delta(G) \geq 2$;

(ii) G has no cut vertex; more generally, deleting any set S of k vertices leaves at most k components;
(iii) if G is bipartite with parts X and Y, then $|X| = |Y|$ (and for a Hamilton *path*, $\big||X| - |Y|\big| \leq 1$).

Sufficient conditions (how to prove "yes" without drawing). Let G be simple with $v \geq 3$ vertices.
(a) **Dirac.** If $\deg(x) \geq v/2$ for every vertex x, then G is Hamiltonian.
(b) **Ore.** If $\deg(x) + \deg(y) \geq v$ for every pair of *non-adjacent* x, y, then G is Hamiltonian.

These are one-way: C_{10} is Hamiltonian but has every degree equal to 2, so $\delta = 2 < 5 = v/2$ and any two non-adjacent vertices have degree sum $4 < 10$: it satisfies neither. When they fail, you must exhibit a cycle or find an obstruction.

Comprehensive example: using the bipartite imbalance

Claim. $K_{3,4}$ has no Hamilton cycle, but it does have a Hamilton path.
No cycle. Suppose C were a Hamilton cycle. Walking along C alternates between the two parts, so the vertices of C split evenly: C would use the same number from each part. But the parts have sizes 3 and 4. Contradiction. ■
A path. With parts $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$, the path $y_1, x_1, y_2, x_2, y_3, x_3, y_4$ visits all seven vertices once. A Hamilton path only needs the parts to differ by at most one, and $4 - 3 = 1$. ✓

"I could not find one" is not a proof

Failing to find a Hamilton cycle proves nothing; you must name an obstruction. Conversely, when you *do* find one, write the cycle out as a vertex sequence beginning and ending at the same vertex, and check two things: every vertex appears exactly once in between, and every consecutive pair really is an edge.

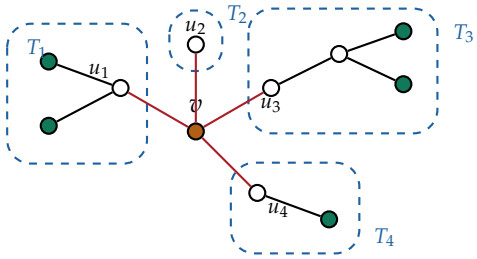
Hamilton in one breath

- Hamilton = every *vertex* once; Euler = every *edge* once. No easy test for Hamilton.
- "Yes": write the cycle and verify it. "No": degree 1, cut vertex, or bipartite imbalance.
- Dirac ($\delta \geq v/2$) and Ore ($\deg x + \deg y \geq v$ for non-adjacent pairs) are sufficient, never necessary.

1.6 Trees and pendant vertices

The intuition

A **tree** is a connected graph with no cycles: the leanest possible connected graph. Tutorial 7 showed that a connected graph on n vertices needs at least $n - 1$ edges; a tree is exactly the case of equality, so removing *any* edge of a tree disconnects it, and adding *any* new edge creates exactly one cycle.
The crucial structural fact is that between any two vertices of a tree there is **exactly one** path. That is why deleting a vertex v of degree k shatters the tree into exactly k pieces, one hanging off each neighbour: there is no second route between the pieces. Every piece, being a finite tree, has a **pendant vertex** (a vertex of degree 1, also called a *leaf*) to spare, and that is the whole idea behind Problem A1.



$\deg(v) = 4$, so $T - v$ has exactly 4 components.

Green dots are pendant vertices of T; u_2 is one as well.

Every component supplies **at least one**, so T has at least 4 pendant vertices.

Figure 7. Deleting a vertex of degree k from a tree leaves exactly k components, each of which contributes a pendant vertex of T.

Simple example: the star and the path

In the star $K_{1,k}$ the centre has degree k and the k leaves are all pendant: exactly k of them, so the bound "at least k" is **tight**. In the path P_n ($n \geq 2$) the maximum degree is 2 and there are exactly 2 pendant vertices, the two ends.

Trees

A **tree** is a connected graph with no cycles. A **pendant vertex** (or **leaf**) is a vertex of degree 1. For a graph T on $n \geq 1$ vertices the following are equivalent:

(i) T is a tree;
(ii) T is connected and $|E(T)| = n - 1$;
(iii) between every two vertices of T there is exactly one path;
(iv) T has no cycle and adding any new edge creates exactly one cycle.

Facts used below.

- Every tree with $n \geq 2$ vertices has at least 2 pendant vertices. *(Take a longest path; neither of its two ends can have a second neighbour, or the path would extend or a cycle would close.)*
- If v is a vertex of a tree T with $\deg(v) = k$, then $T - v$ has exactly k components, one containing each neighbour of v, and each of them is a tree.
- Every subtree of a tree is a tree, and a one-vertex tree has that vertex as a pendant vertex of the bigger tree whenever it was joined to the rest by a single edge.

Comprehensive example: why $T - v$ has exactly k components

Let $\deg(v) = k$ with neighbours $u_1, \ldots, u_k$. Every vertex $x \neq v$ is joined to v by a unique path in T, and that path arrives at v through exactly one u_i; so after deleting v, the vertex x is still joined to that u_i, which puts it in u_i's component. Hence there are *at most* k components. And u_i and u_j ($i \neq j$) are in *different* components: a path from u_i to u_j in $T - v$ together with the edges $u_i v$ and $v u_j$ would be a cycle in T. So there are exactly k. ■

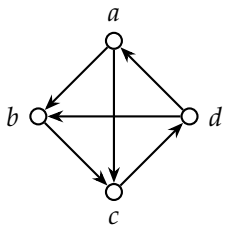
Trees in one breath

- Tree = connected + acyclic = connected with $n - 1$ edges = unique paths.
- $n \geq 2$: at least two pendant vertices. Deleting a degree-k vertex gives exactly k subtrees.

1.7 Orientations and tournaments

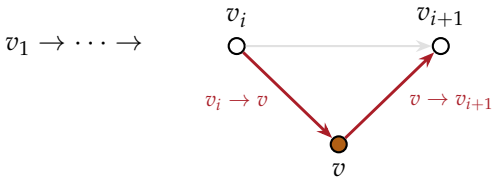
The intuition

To **orient** a graph is to put an arrowhead on each edge, turning it into a directed graph. Each vertex then splits its degree into an in-part and an out-part, and a natural question is whether you can always split it *fairly*, with in and out differing by at most 1. The trick is that walking along an Eulerian circuit and orienting each edge in the direction you travel makes in and out perfectly balanced automatically: every visit to a vertex comes in on one edge and leaves on another. The only obstacle is odd degrees, and Tutorial 7 already showed there is an even number of those, so they can be paired up through one extra helper vertex.
A **tournament** is the opposite extreme: take K_n and orient *every* edge, as if every player had played every other and nobody drew. Rédei's theorem says you can always line up all the players in a row so that each one beat the next, no matter how strange the results were.



a tournament on 4 players

out-degrees (wins): a:2, b:1, c:1, d:2; total $6 = \binom{4}{2}$.
No player beats all the others, and yet both
$a \to b \to c \to d$ and $d \to a \to b \to c$
are directed Hamilton paths.



The old path used the grey arc $v_i \to v_{i+1}$.
Drop v in between: the new path is
$v_1, \ldots, v_i,\ v,\ v_{i+1}, \ldots, v_{n-1}$.

Figure 8. Top: a tournament; the out-degree of a vertex is its number of wins. Bottom: the insertion step in Rédei's proof. As soon as you find a place where the arrows switch from "into v" to "out of v", the new vertex drops into the path there.

Simple example: orienting a triangle and a path

Orient the triangle $a \to b \to c \to a$: every vertex has $\deg^- = \deg^+ = 1$, difference 0. Orient the path $a \to b \to c$: here $\deg^-(a) = 0$, $\deg^+(a) = 1$ (difference 1), $\deg^-(b) = \deg^+(b) = 1$ (difference 0), $\deg^-(c) = 1$, $\deg^+(c) = 0$ (difference 1). The two odd-degree vertices are exactly the two with a difference of 1, which is no accident: see Problem A3.

Orientation, tournament, Rédei's theorem

An **orientation** of a graph G is a directed graph obtained by replacing each edge $\{x, y\}$ by exactly one of the arcs (x, y) or (y, x). A **tournament** on n vertices is an orientation of K_n: for every pair of distinct vertices exactly one of (x, y), (y, x) is an arc.
A **directed Hamilton path** in a digraph is a sequence $w_1, w_2, \ldots, w_n$ listing **every** vertex exactly once, with $w_i \to w_{i+1}$ an arc for $i = 1, \ldots, n-1$.
Rédei's theorem (1934). Every finite tournament has a directed Hamilton path.

Camion's theorem (1959). Every *strongly connected* tournament has a directed Hamilton *cycle*. (Strong connectedness is needed: in the tournament where player 1 beats everybody, no cycle can return to player 1.)

Comprehensive example: the ranking has to be local, not global

In a tournament, Rédei's theorem does **not** say there is a "best player". The 3-cycle $a \to b \to c \to a$ has no undefeated player, and its directed Hamilton paths are a, b, c and b, c, a and c, a, b. What the theorem promises is only a chain in which each player beats the *next one*, and such a chain tells you nothing about whether the first player beats the last. This is the standard warning about transitivity in sports rankings, and it is why the proof builds the chain by *insertion* rather than by sorting.

Directed graphs in one breath

- Orientation: one arrowhead per edge. Travelling an Eulerian circuit and orienting as you go balances every vertex exactly.
- Tournament = oriented K_n; out-degree = wins; Rédei: a directed Hamilton path always exists.

The toolbox at a glance

Situation	Tool	Signal words
Count regions of a drawing	$v - e + r = 2$; $\sum_R \deg(R) = 2e$	"faces", "regions", "polyhedron"
Prove *not* planar, fast	$e > 3v - 6$, or $e > 2v - 4$ if triangle-free	"is it planar", "too many edges"
Prove *not* planar, bound holds	Kuratowski: subgraph homeomorphic to K_5 or $K_{3,3}$	"Kuratowski", "Petersen"
Prove planar	exhibit one crossing-free drawing	"draw", "embedding"
Small degree guaranteed	average degree < 6, so $\delta \leq 5$	"at most 5", "colouring"
Non-edges become edges	$\overline{G}$; $\overline{K_{m,n}} = K_m \cup K_n$	"complement"
Every vertex once	Hamilton: exhibit, or degree 1 / cut vertex / imbalance	"Hamilton", "round table", "tour"
Every edge once	Euler: connected + count odd vertices	"Eulerian", "bridges"
Acyclic connected	tree: $n - 1$ edges, unique paths, ≥ 2 leaves	"tree", "pendant"
Arrowheads	orient along an Eulerian circuit	"orient", "in-degree", "tournament"

2 The tutorial problems, worked in full

This section solves the ten problems of the *Graphs Tutorial 3* sheet, in the order they appear there. Section 3 then solves the five problems of the *Additional* sheet.

2.1 How to write the solutions

Three kinds of answer come up this week, and each has its own shape.

Writing template 1: "is this graph planar?"

Statement. *[Name the graph; record v and e.]*
Try to rule it out. *[Is it simple with $v \geq 3$? Then $e \leq 3v - 6$ must hold. Triangle-free (for example bipartite)? Then $e \leq 2v - 4$ must hold. If a bound fails, you are done: say "contradiction, so G is not planar".]*
If no bound fails. *[The bounds decide nothing. Either **draw** a crossing-free embedding, which proves planar, or **find** a subgraph homeomorphic to K_5 or $K_{3,3}$, which proves non-planar by Kuratowski's theorem.]*
Conclusion. *["Hence G is / is not planar." ■]*

Writing template 2: "exhibit a subgraph homeomorphic to $K_{3,3}$"

The subgraph. *["Delete the edges … from G. Call the result H." Deleting edges is legitimate: a subgraph of a planar graph is planar, so if H is non-planar then so is G.]*
The branch vertices. *["Take $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$." These must be the degree-3 vertices of H.]*
The nine paths. *[List one path $x_i \to y_j$ for each of the nine pairs, and state that their **interior vertices are distinct** so the paths are internally disjoint.]*
Conclusion. *["So H is a subdivision of $K_{3,3}$, hence homeomorphic to $K_{3,3}$; by Kuratowski's theorem G is non-planar." ■]*

Writing template 3: "does a Hamilton cycle exist?"

Yes case. *[Write the cycle as a vertex sequence starting and ending at the same vertex. Then verify: (1) every other vertex appears exactly once; (2) every consecutive pair is an edge.]*
No case. *[Name one obstruction and finish it: a vertex of degree 1; a cut vertex c ("deleting c leaves 2 components, but deleting one vertex of a cycle leaves a path, which is connected"); a bipartite imbalance ("a cycle alternates between the parts, so the parts have equal size, but $|X| \neq |Y|$").]*
Conclusion. *["Hence G is / is not Hamiltonian." ■]*

Phrasebook: new sentences for this tutorial

Where	What to write
Euler's formula	"G is connected and planar, so every embedding has $r = e - v + 2$ regions."
Region degrees	"Each region has at least 3 edges on its boundary, and each edge borders two region-sides, so $3r \leq 2e$."
Edge bound fails	"If G were planar then $e \leq 3v - 6$; but $e = \cdots > \ldots$, a contradiction."
Edge bound holds	"The bound is satisfied, so it decides nothing; we exhibit an embedding instead."
Subgraph argument	"Every subgraph of a planar graph is planar, so it suffices to find one non-planar subgraph."
Kuratowski	"H is a subdivision of $K_{3,3}$ with branch vertices …, so by Kuratowski's Theorem G is non-planar."
Minimum vs average	"The minimum degree is at most the average degree, which is $2e/v < 6$."
Hamilton, yes	"The sequence … visits every vertex exactly once and consecutive vertices are adjacent."
Hamilton, no	"A Hamilton cycle uses exactly two edges at each vertex, but $\deg(z) = 1$."
Bipartite by parity	"Every edge joins a string of even weight to a string of odd weight."
Eulerian repair	"Each new edge changes the parity of exactly two degrees, so it removes at most 2 odd vertices."

2.2 P1: the octahedron

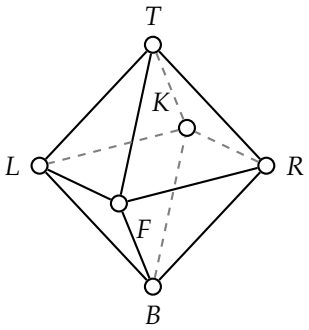
Problem P1

An octahedron is a regular polyhedron made up of 8 equilateral triangles (it sort of looks like two pyramids with their bases glued together). Draw a planar graph representation of an octahedron. How many vertices, edges and faces does an octahedron (and your graph) have?

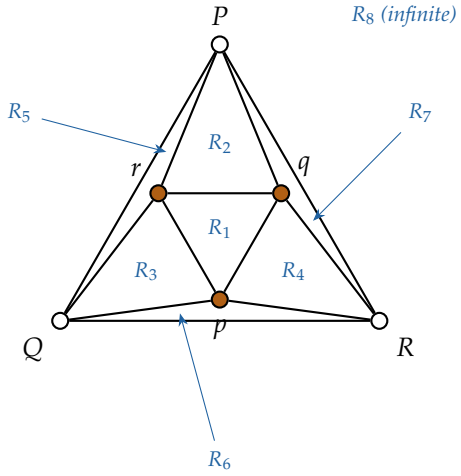
How to think about it. Build the solid first and read the numbers off it: two pyramids on a shared square base. Then flatten it. The standard way to flatten a convex polyhedron is to imagine it made of elastic, poke a hole through one face, and stretch that face open until the whole solid lies flat inside it. The face you poke through becomes the **infinite region**, which is why the drawing has one region for every face of the solid.

Model solution to P1

Solution (what you write)	Commentary (what it does)
The solid. Two square pyramids share a square base. Vertices: the 4 corners of the base plus the two apexes, so $v = 6$. Faces: 4 triangles on each pyramid, so $r = 8$, as the problem says. Edges: the 4 base edges plus 4 from each apex to the base, so $e = 4 + 4 + 4 = 12$.	*Count the solid directly; do not guess.*
Check with the Handshaking Theorem. Every vertex of the octahedron has degree 4 (each apex meets the 4 base corners; each base corner meets 2 base neighbours and the 2 apexes). So $\sum \deg = 6 \cdot 4 = 24 = 2 \cdot 12$. ✓	*A free consistency check, always worth two lines.*
Check with Euler's formula. $v - e + r = 6 - 12 + 8 = 2$. ✓	*Euler's formula is the second free check.*



the solid: two square pyramids glued along $LFRK$



the planar embedding: a triangle inside a triangle

Figure 9. The octahedron and one planar embedding. Non-adjacent ("antipodal") pairs are $P-p$, $Q-q$, $R-r$, so the octahedron graph is the complete tripartite graph $K_{2,2,2}$. All eight regions are triangles.

The planar drawing. Draw an outer triangle PQR and a smaller triangle pqr inside it, rotated so that each inner vertex faces an outer edge. Then join

$$p-Q, \quad p-R, \quad q-P, \quad q-R, \quad r-P, \quad r-Q.$$

No two of these twelve edges cross (Figure 9), so this is a planar embedding.

Each inner vertex is joined to the two outer vertices nearest it, so the six new edges stay inside their own corner of the picture.

Reading the drawing. The embedding has $v = 6$ and $e = 12$, and its regions are

- the inner triangle $p\,q\,r$ (1)
- one triangle in each corner, between an outer vertex and the inner triangle: $P\,q\,r$, $Q\,r\,p$, $R\,p\,q$ (3)
- one triangle on each outer edge: $P\,r\,Q$, $Q\,p\,R$, $R\,q\,P$ (3)
- the infinite region, outside the triangle PQR (1)

which is $1+3+3+1 = 8$ regions, exactly the 8 faces of the solid.

The infinite region is the face you "poked through". Every region is a triangle, matching the 8 equilateral triangles of the octahedron.

Answer. $\boxed{v = 6, \quad e = 12, \quad r = 8}$ for the octahedron and for the graph.

Two remarks worth knowing. (i) Every vertex is adjacent to all others except its antipode, so the graph is $\overline{3K_2} = K_{2,2,2}$, and it is 4-regular. (ii) $e = 12 = 3 \cdot 6 - 6$, so the octahedron **attains** the planar edge bound: it is a *triangulation*, and no graph on 6 vertices can be planar with 13 edges.

Section 1.2: all regions triangles $\iff e = 3v - 6$.

Another way to see it

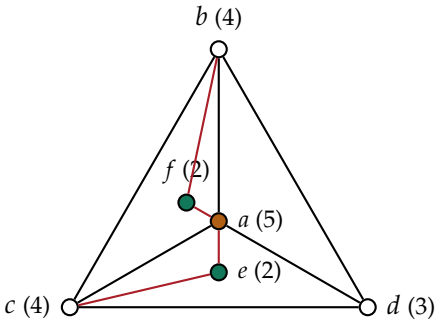
If you already believe the drawing is planar, you do not have to count the regions by hand at all: Euler's formula does it for you, $r = e - v + 2 = 12 - 6 + 2 = 8$. Counting them anyway, as above, is a check on the drawing, and it is how you catch a drawing with an accidental crossing.

2.3 P2: could a graph with degrees $2, 2, 3, 4, 4, 5$ be planar?

Problem P2

The graph G has 6 vertices with degrees $2, 2, 3, 4, 4, 5$. Could G be planar?

How to think about it. Start with the arithmetic: the degree sequence gives e through the Handshaking Theorem, and then you can test the edge bound. Here the bound will *hold*, which proves nothing (Section 1.2), so the honest answer needs a drawing. Build the graph from the degree sequence: the vertex of degree 5 must be joined to *all* the others, so the graph is "one vertex joined to everything, plus whatever is left".



a planar embedding of G

$\sum \deg = 2 + 2 + 3 + 4 + 4 + 5 = 20$, so $e = 10$.

Bound: $3v - 6 = 3 \cdot 6 - 6 = 12 \geq 10$. *It holds,* so the bound decides **nothing**: we must draw.

$\{a, b, c, d\}$ is a K_4; f hangs in the region abc and e in the region acd, so nothing crosses.

Regions: $r = e - v + 2 = 10 - 6 + 2 = 6$.

*Figure 10. Up to isomorphism this is the **only** graph with degree sequence $(5, 4, 4, 3, 2, 2)$, and it is planar. Red edges are the two degree-2 vertices, each tucked inside a region of the K_4 on $\{a, b, c, d\}$.*

Model solution to P2

Solution (what you write)	**Commentary (what it does)**
Step 1: the number of edges. By the Handshaking Theorem, $2e = \sum_x \deg(x) = 2 + 2 + 3 + 4 + 4 + 5 = 20$, so $e = 10$. (Good: the sum is even, as it must be, and there are two odd degrees, 3 and 5, an even number of them.)	*Always sanity-check the parity before anything else.*
Step 2: what the edge bound says. G is simple with $v = 6 \geq 3$, so if G is planar then $e \leq 3v - 6 = 12$. Here $e = 10 \leq 12$: the bound is **satisfied**.	*Careful: this does* not *prove planarity. It only fails to disprove it.*

Step 3: build the graph. Name the vertices a, b, c, d, e, f with $\deg a = 5$, $\deg b = \deg c = 4$, $\deg d = 3$, $\deg e = \deg f = 2$. Since $\deg(a) = 5 = v - 1$, the vertex a is adjacent to all five others. Removing a lowers every other degree by exactly 1, leaving a graph H on $\{b, c, d, e, f\}$ with degrees $3, 3, 2, 1, 1$ and $10 - 5 = 5$ edges. The only such graph is the triangle bcd with one pendant vertex at b and one at c.

Removing a dominating vertex is the standard way to pin down a degree sequence.

Step 4: a drawing. Note that $\{a, b, c, d\}$ carries all six of its pairs, so it is a K_4; draw it as a triangle bcd with a at the centre. This splits the inside of the triangle into the three regions abc, acd, abd. Place f inside the region abc and join it to a and b; place e inside the region acd and join it to a and c. No edge crosses another (Figure 10).

A pendant pair can always be tucked into a region that contains both of its neighbours on its boundary.

Answer. Yes: G is planar and in fact *must* be, since up to isomorphism this degree sequence has only this one graph. Its embedding has $r = 10 - 6 + 2 = 6$ regions.

The point of the problem. A degree sequence can **rule out** planarity, when $\sum \deg > 2(3v - 6)$, but it can never **confirm** it on its own: you still need a drawing or a Kuratowski subgraph. Compare $K_{3,3}$, whose degrees $3, 3, 3, 3, 3, 3$ also pass the $3v - 6$ test, yet which is not planar.

Same v, fewer edges, and still non-planar: the count is not the whole story.

A tempting wrong answer

"$e = 10 \leq 12 = 3v - 6$, so G is planar." This is the single most common error of the whole tutorial. The edge bound is a **necessary** condition, not a sufficient one. The conclusion happens to be true here, but the argument does not establish it, and the same argument applied to $K_{3,3}$ gives a false conclusion.

2.4 P3: every planar graph has a vertex of degree at most 5

Problem P3

Prove that any planar graph has a vertex of degree at most 5. *(Hint: $|E(G)| \leq 3|V(G)| - 6$.)*

How to think about it. The hint caps the number of edges, and the Handshaking Theorem turns a cap on edges into a cap on the *average* degree. The minimum of a list of numbers is never larger than its average, so a cap on the average is a cap on the minimum. The only care needed is the small cases, where the hint does not apply.

Model solution to P3

Solution (what you write)	Commentary (what it does)
Statement. Let $G = (V, E)$ be a **simple** planar graph (no loops, no multiple edges) with at least one vertex. Then $\delta(G) \leq 5$, that is, G has a vertex of degree at most 5.	*Say "simple": with loops the claim is false, since a vertex carrying three loops has degree 6.*

Small cases ($v \leq 2$). Every degree is at most $v - 1 \leq 1 \leq 5$, so the claim holds.

The bound $e \leq 3v - 6$ is only available for $v \geq 3$, so these must be handled separately.

Main case ($v \geq 3$). Then $e \leq 3v - 6$. *Why this holds even if G is disconnected:* add edges joining different components, drawing each new edge inside a region that touches both; this keeps the graph planar and simple and only increases e, so the bound for the connected graph we end up with also bounds the original e.

Most textbook statements assume connected; say one sentence about why that is not a restriction.

Averaging. Let $\bar{d}$ be the average degree. By the Handshaking Theorem,

$$\delta(G) \leq \bar{d} = \frac{\sum_{x \in V} \deg(x)}{v} = \frac{2e}{v} \leq \frac{2(3v-6)}{v} = 6 - \frac{12}{v} < 6.$$

The minimum of a list is at most its average; this is the only inequality that matters.

Conclusion. So $\delta(G) < 6$, and since $\delta(G)$ is an integer, $\delta(G) \leq 5$: some vertex of G has degree at most 5. ■

"< 6 and an integer" is what gives ≤ 5; do not skip this half-line.

Another way to see it, and why the result matters

A contradiction version. Suppose every vertex had degree ≥ 6. Then $2e = \sum \deg(x) \geq 6v$, so $e \geq 3v > 3v - 6$, contradicting the bound. Same inequality, written backwards; use whichever you find clearer.

Why 5 and not 4. The icosahedron is planar, 5-regular, with $v = 12$, $e = 30 = 3 \cdot 12 - 6$. So there is no planar graph guarantee below 5: the result is sharp.

What it is for. This lemma is the engine of graph colouring. Pick a vertex x with $\deg(x) \leq 5$, colour $G - x$ by induction with 6 colours, then put x back: its at most 5 neighbours use at most 5 colours, so a sixth is free. That proves the **Six Colour Theorem**, and refining the same idea gives the Five Colour Theorem.

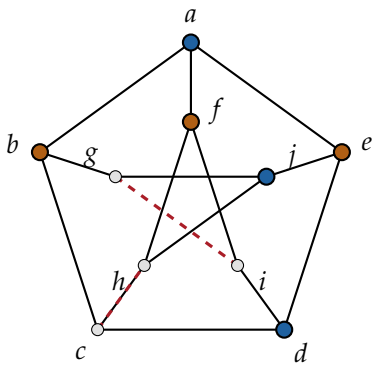
2.5 P4: the Petersen graph is not planar (Kuratowski, via $K_{3,3}$)

Problem P4

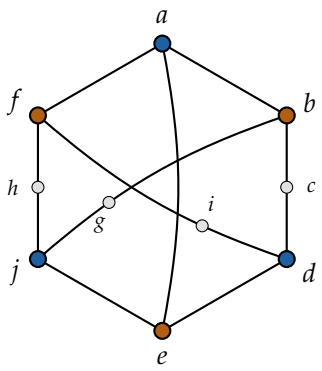
Show that the Petersen graph is not planar by the Kuratowski theorem. *(Hint: use $K_{3,3}$.)*

How to think about it. First check why the easy route is closed. The Petersen graph has $v = 10$ and $e = 15$, and $3v - 6 = 24 \geq 15$: the general bound holds. It is not bipartite (it has 5-cycles), so the triangle-free bound $2v - 4 = 16 \geq 15$ also holds. So counting edges will not do it, and we must find a Kuratowski subgraph. Moreover the Petersen graph is 3-regular, so it cannot contain a subdivision of K_5 (which needs five vertices of degree *at least* 4, and every Petersen vertex has degree 3); $K_{3,3}$ is the only option, exactly as the hint says.

The search itself: a subdivision of $K_{3,3}$ needs six branch vertices of degree 3. Every vertex of the Petersen graph already has degree 3, so we should **delete** edges until exactly six vertices of degree 3 are left. Deleting two edges kills the degree of four vertices down to 2, leaving 6 vertices of degree 3, which is exactly what we need, and those four degree-2 vertices become the subdivision points.



the Petersen graph; dashed red are the two deleted edges ch and gi



the same subgraph redrawn: a subdivision of $K_{3,3}$

Figure 11. Deleting the two edges $\{c,h\}$ and $\{g,i\}$ from the Petersen graph leaves a subdivision of $K_{3,3}$ with branch vertices $X = \{a,d,j\}$ (blue) and $Y = \{b,e,f\}$ (orange). The four grey vertices c,g,h,i are now degree-2 subdivision points, one inside each of the four paths $d-b$, $j-f$, $d-f$, $j-b$. In the right-hand drawing the six branch vertices sit on a hexagon and the three remaining edges are its long diagonals; the crossings are unavoidable, which is the whole point.

Model solution to P4

Solution (what you write)	**Commentary (what it does)**
Set-up. Label the Petersen graph as usual: outer 5-cycle a,b,c,d,e; inner pentagram f,g,h,i,j with edges fh, hj, jg, gi, if; and five spokes af, bg, ch, di, ej. It has $v = 10$, $e = 15$, and it is 3-regular.	*Write the edge list once, explicitly; every later claim is checked against it.*
Why the counting arguments fail. $3v - 6 = 24 > 15 = e$, and the Petersen graph has no triangle and no 4-cycle (its girth is 5), so even the triangle-free bound $2v - 4 = 16 > 15$ holds. No edge count can settle this, so we use Kuratowski's theorem.	*Saying* why *the easy route is closed is part of the answer.*
The subgraph. Let H be the Petersen graph with the two edges $\{c,h\}$ and $\{g,i\}$ deleted. Then H is a subgraph of the Petersen graph, and every subgraph of a planar graph is planar; so it is enough to show that H is non-planar.	*Deleting edges can only help: it never makes a planar graph non-planar.*
Degrees in H. Deleting those two edges lowers the degree of c,h,g,i to 2 and leaves the other six vertices at degree 3: $\deg_H = 3$ for a,b,d,e,f,j; $\deg_H = 2$ for c,g,h,i.	*Exactly six branch vertices and four subdivision points: the right shape for $K_{3,3}$.*
The branch vertices. Put $X = \{a,d,j\}$ and $Y = \{b,e,f\}$.	*Three and three; now produce all nine cross connections.*

The nine internally disjoint paths. In H:

$a-b$:	a, b	(edge of the outer cycle)
$a-e$:	a, e	(edge of the outer cycle)
$a-f$:	a, f	(spoke)
$d-b$:	d, c, b	(outer edges dc, cb)
$d-e$:	d, e	(edge of the outer cycle)
$d-f$:	d, i, f	(spoke di, pentagram edge if)
$j-b$:	j, g, b	(pentagram edge jg, spoke gb)
$j-e$:	j, e	(spoke ej)
$j-f$:	j, h, f	(pentagram edges jh, hf)

Check both: (1) every listed pair really is an edge of the Petersen graph; (2) the interior vertices are all different. These are the two places marks are lost.

Five of the nine are single edges; the other four have exactly one interior vertex, namely c, i, g, h respectively. These four are **distinct**, and none of them lies in $X \cup Y$, so the nine paths are internally disjoint. Together they use all 13 edges of H.

Conclusion. H is obtained from $K_{3,3}$ (with parts X and Y) by four elementary subdivisions, one on each of the edges db, df, jb, jf. Hence H is homeomorphic to $K_{3,3}$, and since $K_{3,3}$ is non-planar, so is H. By Kuratowski's Theorem, the Petersen graph contains a subgraph homeomorphic to $K_{3,3}$ and is therefore **non-planar**. ■

Figure 11 redraws H so that the $K_{3,3}$ pattern is visible.

Another way to see it: the girth bound

The Petersen graph has **girth** 5 (no triangle, no 4-cycle), and for a planar graph in which every cycle has length at least k the edge bound sharpens to $e \leq \frac{k}{k-2}(v-2)$. With $k = 5$ and $v = 10$,

$$e \;\leq\; \tfrac{5}{3}\,(10-2) \;=\; \tfrac{40}{3} \;=\; 13\tfrac{1}{3},$$

so a planar graph of girth 5 on 10 vertices has at most 13 edges. The Petersen graph has 15. Contradiction, so it is not planar. This is shorter, but the problem asks specifically for Kuratowski, so give the argument above and offer this as a remark.

Two things to get right in a Kuratowski answer

(i) Internally disjoint. The nine paths may share their *endpoints* (that is the point: each branch vertex is on three of them) but must share no *interior* vertex. Without that, what you have drawn is not a subdivision.

(ii) It is a subgraph, not the whole graph. You are allowed to throw edges away, and here you must: the Petersen graph itself has ten vertices of degree 3, which is four too many.

2.6 P5: the complement of $K_{m,n}$

Problem P5

What is the complement of the complete bipartite graph $K_{m,n}$? Does it contain a path of length n? A path of length m? What about cycles?

How to think about it. Write down what "edge of the complement" means in words: a pair of vertices that was *not* joined in $K_{m,n}$. In $K_{m,n}$ the pairs that are not joined are exactly the pairs lying inside the same part. So the complement has all the edges inside the parts and none between them, which is two complete graphs sitting side by side. After that, everything is about one simple fact: a path or a cycle is connected, so it cannot jump from one part to the other.

Model solution to P5

Solution (what you write)	**Commentary (what it does)**
Set-up. Let $K_{m,n}$ have parts V_1 with $\lvert V_1\rvert = m$ and V_2 with $\lvert V_2\rvert = n$, so $v = m+n$. Its edges are exactly the pairs with one end in each part.	*Fix notation before computing.*
The complement. A pair $\{x, y\}$ is an edge of $\overline{K_{m,n}}$ exactly when it is *not* an edge of $K_{m,n}$, that is, when x and y lie in the **same** part. So $\overline{K_{m,n}}$ consists of a complete graph on V_1, a complete graph on V_2, and no edge between them: $\boxed{\overline{K_{m,n}} = K_m \cup K_n}$ (disjoint union).	*Figure 5 shows the case $m = 2$, $n = 3$.*
Check the edge count. $\binom{m}{2} + \binom{n}{2} + mn = \binom{m+n}{2}$, as it must be, since the two graphs share out the edges of K_{m+n}.	*A one-line identity check that catches sign errors.*
The key observation. A path and a cycle are **connected** subgraphs, so each one lies entirely inside a single connected component. The components here are K_m and K_n. Inside K_t: the longest path has t vertices and so length $t-1$, and paths of *every* length $0, 1, \ldots, t-1$ occur; the longest cycle has length t (for $t \geq 3$), and cycles of *every* length $3, 4, \ldots, t$ occur. For $t \leq 2$ there is no cycle at all.	*"Length = number of edges", so a path of length ℓ needs $\ell+1$ vertices.*
A path of length n? It must fit in K_m or in K_n, so it needs $n+1 \leq m$ or $n+1 \leq n$. The second is impossible. Hence $\overline{K_{m,n}}$ contains a path of length $n \iff \boxed{m \geq n+1}$.	*Count vertices, not edges: a path of length n has $n+1$ vertices.*
A path of length m? By the same argument with the roles swapped, $\overline{K_{m,n}}$ contains a path of length $m \iff \boxed{n \geq m+1}$.	*The two answers are mirror images; say so, it shows you understand the structure.*
In particular **at most one** of the two can happen, and when $m = n$ **neither** does: the longest path in $\overline{K_{n,n}} = K_n \cup K_n$ has length $n-1$.	

Cycles. $\overline{K_{m,n}}$ has a cycle if and only if $\max(m,n) \geq 3$, and then it has cycles of exactly the lengths

$$3, 4, \ldots, \max(m,n),$$

the longest one being a Hamilton cycle of the larger component. For $m, n \geq 1$ it is never Hamiltonian as a whole graph, because it has two components. (The degenerate case $n = 0$ is different: $\overline{K_{m,0}} = K_m$ is connected, and Hamiltonian exactly when $m \geq 3$.)

K_1 and K_2 contain no cycle, which is why $\max(m,n) \geq 3$ is needed.

Worked instance. $m = 2$, $n = 5$: $\overline{K_{2,5}} = K_2 \cup K_5$. Path of length $n = 5$? Needs 6 vertices in one component; the biggest component has 5. **No**. Path of length $m = 2$? Needs 3 vertices in one component, and K_5 has 5. **Yes**. Cycles: lengths 3, 4, 5. ✓

Always run the general answer on one concrete pair.

P5 in one line

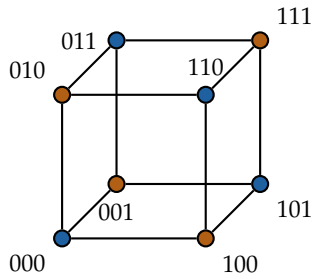
$\overline{K_{m,n}} = K_m \cup K_n$; longest path $\max(m,n) - 1$; longest cycle $\max(m,n)$ when that is at least 3; and, for $m, n \geq 1$, a path of length n exists iff $m > n$ and a path of length m iff $n > m$.

2.7 P6: the hypercube is bipartite

Problem P6

Prove that the hypercube is bipartite.

How to think about it. To prove a graph bipartite you must **produce** the two parts and then check that no edge stays inside one of them. The hypercube hands you a rule for free: its vertices are bit strings, and its edges change exactly one bit. Changing one bit changes the number of 1s by exactly 1, so it flips the **parity** of that number. Parity is therefore the colouring, and the proof is three lines.



Q_3 coloured by parity of weight

- even weight: 000, 110, 101, 011
- odd weight: 100, 010, 001, 111

Every edge flips one bit, so it changes the weight by exactly ± 1 and always joins a blue vertex to an orange one.

Consequence: Q_n has no odd cycle, so no triangle and no 5-cycle, for any n.

Figure 12. The parity 2-colouring of Q_3. The same rule works for every n, which is why $Q_n = K_2 \times \cdots$ is bipartite with parts of size 2^{n-1} each.

Model solution to P6

Solution (what you write)	**Commentary (what it does)**

Set-up. For $n \geq 1$ the **hypercube** Q_n has vertex set $\{0,1\}^n$, the binary strings of length n, and two strings are adjacent exactly when they **differ in exactly one coordinate**. For a string $x = x_1 x_2 \cdots x_n$ define its **weight**

Give the definition you are going to use; the weight is the whole proof.

$$w(x) = x_1 + x_2 + \cdots + x_n = \#\{\, i : x_i = 1 \,\}.$$

The two parts. Put

State both halves of "partition": covers everything, overlaps nowhere.

$$V_1 = \{\, x \in \{0,1\}^n : w(x) \text{ is even} \,\},$$
$$V_2 = \{\, x \in \{0,1\}^n : w(x) \text{ is odd} \,\}.$$

Every string has a weight, and that weight is either even or odd but not both, so $V_1 \cup V_2 = V(Q_n)$ and $V_1 \cap V_2 = \emptyset$: this is a partition.

Every edge crosses. Let $\{x, y\}$ be an edge of Q_n. Then x and y differ in exactly one coordinate, say coordinate i, and agree everywhere else. So

This is the only step with content. "Differ by 1" forces opposite parities.

$$w(y) = w(x) + 1 \quad \text{if } x_i = 0,\ y_i = 1,$$
$$w(y) = w(x) - 1 \quad \text{if } x_i = 1,\ y_i = 0.$$

In both cases $w(x)$ and $w(y)$ differ by 1, so one is even and the other is odd. Hence one of x, y lies in V_1 and the other in V_2: no edge has both ends in the same part.

Conclusion. $V(Q_n) = V_1 \cup V_2$ is a partition with every edge joining V_1 to V_2, so Q_n is **bipartite** for every $n \geq 1$. ■

The balanced-parts fact is the one you need for Hamilton questions about Q_n.

Two free corollaries. (i) Q_n has no cycle of odd length, so no triangle: a cycle must alternate between V_1 and V_2 and therefore returns to its start only after an even number of steps. (ii) The two parts have the same size, $|V_1| = |V_2| = 2^{n-1}$, since $\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$. So Q_n is not ruled out as Hamiltonian by the imbalance test, and indeed it is Hamiltonian for every $n \geq 2$ (a *Gray code*).

Check on Q_3. $V_1 = \{000, 110, 101, 011\}$, $V_2 = \{100, 010, 001, 111\}$, four and four, and all 12 edges run between them (Figure 12). ✓

Say which bit, not just "they are close"

"x and y are adjacent so they are almost the same, hence different parities" is not an argument. Write down the coordinate that changes and conclude $|w(x) - w(y)| = 1$. The whole proof hinges on that single equation.

2.8 P7: the Königsberg bridge builder

Problem P7

A bridge builder has come to Königsberg and would like to add bridges so that it is possible to travel over every bridge exactly once. How many bridges must be built?

How to think about it. "Travel over every bridge exactly once" is an **Eulerian trail** (you are not asked to come back to where you started, so a tour is not required). Euler's theorem says a connected graph has an Eulerian trail exactly when it has 0 or 2 vertices of odd degree. Königsberg has 4, so the question becomes: how few edges can I add to bring the number of odd vertices down to 2? Each new edge touches two vertices and flips the parity of each, so one edge can repair at most two odd vertices. That gives the lower bound, and one well-chosen bridge gives the matching construction.

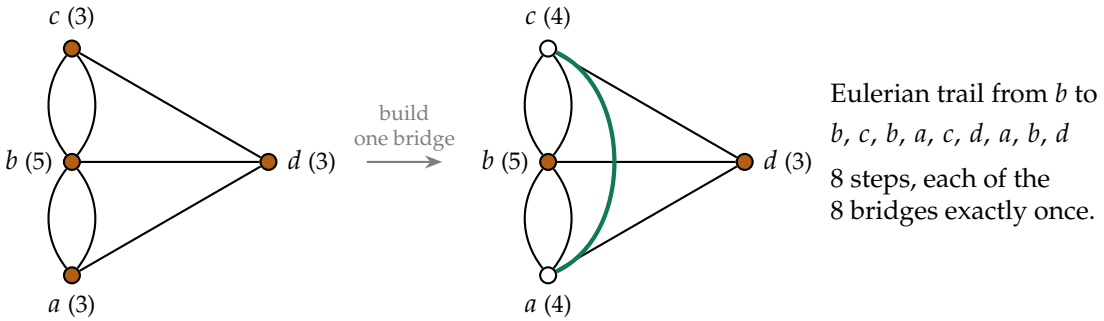


Figure 13. Orange vertices have odd degree. One extra bridge between two of the four odd land masses leaves exactly two odd vertices, which is what an Eulerian trail needs.

Model solution to P7

Solution (what you write)	Commentary (what it does)
The model. Land masses are vertices and bridges are edges. Königsberg is the **multigraph** G on $\{a,b,c,d\}$ with two $a-b$ bridges, two $b-c$ bridges, and one each of $c-d$, $b-d$, $a-d$: seven edges in all. Its degrees are $$\deg a = 3, \quad \deg b = 5, \quad \deg c = 3, \quad \deg d = 3,$$ and $3+5+3+3 = 14 = 2\cdot 7$. ✓ The graph is connected.	*Multiple edges are fine: Euler's theorem is about degrees and connectedness only.*
Restating the goal. A route that uses every bridge exactly once is an **Eulerian trail**. By Euler's theorem, a connected graph has one exactly when the number of odd-degree vertices is 0 (then it is a closed tour) or 2 (then it runs between the two odd vertices). Here **all four** vertices are odd, so at present no such route exists: that was Euler's 1736 answer.	*Name the theorem and the two allowed cases.*

Lower bound: one bridge is necessary. Adding a new bridge $\{x, y\}$ increases $\deg x$ and $\deg y$ by 1 each, so it flips the parity of exactly two vertices. Hence each new bridge changes the number of odd-degree vertices by $+2$, 0 or -2: it can remove at most two of them. We start with 4 odd vertices and need at most 2, so we must remove at least 2, and therefore must build at least $\lceil (4-2)/2 \rceil = 1$ bridge.

This parity-counting sentence is the heart of the answer; it also gives the general formula below.

Upper bound: one bridge is enough. Build a single new bridge between two of the odd land masses, say between a and c. Now $\deg a = 4$, $\deg c = 4$, $\deg b = 5$, $\deg d = 3$: exactly **two** odd vertices, b and d, and the graph is still connected. By Euler's theorem there is an Eulerian trail from b to d.

Join two odd *vertices; joining two even ones would make things worse.*

An explicit route on the repaired graph, writing $b-c_1, b-c_2$ for the two bc bridges and $a-b_1, a-b_2$ for the two ab bridges:

Always write one explicit trail: it proves the construction works.

$$b \xrightarrow{bc_1} c \xrightarrow{bc_2} b \xrightarrow{ab_1} a \xrightarrow{ac\ (\text{new})} c \xrightarrow{cd} d \xrightarrow{ad} a \xrightarrow{ab_2} b \xrightarrow{bd} d.$$

That is 8 steps using each of the 8 bridges exactly once. ✓

Answer. $\boxed{1 \text{ bridge}}$ is necessary and sufficient, and the walk cannot be closed: it must start at one of b, d and finish at the other.

If the builder also wants to finish where he started (an Eulerian **tour**, 0 odd vertices), then he must remove all 4 odd vertices, so he needs at least $4/2 = 2$ bridges, and 2 suffice: build $a-c$ and $b-d$, making all degrees $4, 6, 4, 4$. **General rule:** a connected (multi)graph with $2k$ odd vertices needs $k-1$ extra edges for a trail and k for a tour.

Watch for this variant on an exam; it is the same count with a different target.

Trail or tour?

"Travel over every bridge exactly once" does **not** ask you to return home, so two odd vertices are allowed and the answer is 1. If the wording had been "… and return to your starting point", the answer would be 2. Read the question twice and say which of the two you are solving.

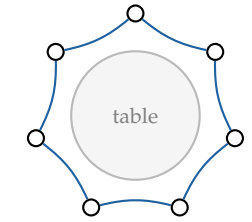
2.9 P8: seating a group of friends round a table

Problem P8

A graph represents friendships between a group of students (each vertex is a student, each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends? What does this question have to do with paths?

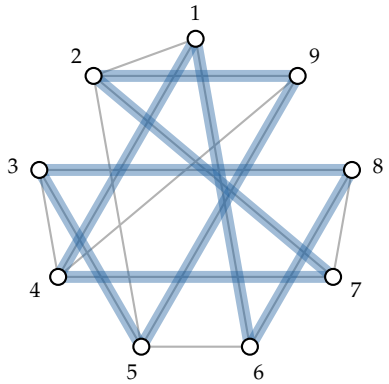
How to think about it. Translate first, compute second. A round table of k students is a cyclic order $s_1, s_2, \ldots, s_k, s_1$: each student has exactly two neighbours at the table, the one on the left and the one on the right. "Every student sits between two friends" says that both of those pairs are edges, for every student. So a valid seating is exactly a **closed walk that visits every vertex exactly once**, that is, a **Hamilton cycle**. The answer to the second half of the question is therefore: it is the Hamilton

question, the "every vertex once" twin of the Euler "every edge once" question, and a Hamilton cycle is nothing but a Hamilton *path* whose two ends happen to be joined by an edge.



a seating of 7 students

$\Longleftrightarrow$ same thing



a friendship graph on 9 students;
blue: the seating $1, 4, 7, 2, 9, 5, 3, 8, 6, 1$

Figure 14. Left: a round table is a cycle through all the people. Right: a worked example. The grey graph is the friendships; the blue edges are a Hamilton cycle, that is, a valid seating. Note that the seating is ***not*** *the order* $1, 2, \ldots, 9$ *in which the students happen to be drawn.*

Model solution to P8

Solution (what you write)	**Commentary (what it does)**
The translation. Seat the students around the table and read off the cyclic order $s_1, s_2, \ldots, s_k, s_1$ where k is the number of students. Student s_i sits between s_{i-1} and s_{i+1} (indices modulo k), so the condition "every student sits between two friends" says precisely that $\{s_i, s_{i+1}\}$ is an edge for every i. Since each student appears exactly once at the table, the sequence $s_1, s_2, \ldots, s_k, s_1$ is a cycle that visits every vertex exactly once: a **Hamilton cycle**. Conversely every Hamilton cycle gives a valid seating by reading it round the table.	*Prove both directions of the equivalence; one sentence each is enough.*
valid seating $\Longleftrightarrow$ Hamilton cycle.	
The connection with paths. A Hamilton **path** visits every vertex exactly once *in a row*; a Hamilton cycle is a Hamilton path $s_1, \ldots, s_k$ whose two ends are also friends, so the line can be bent into a ring. So the seating question is the **Hamilton path / cycle** question, which is the vertex-version of the Euler trail / tour question ("every edge once") from Tutorial 7. Seating the students in a *row*, so that each student only has to be friends with the person next to them, is exactly asking for a Hamilton path.	*This is the part of the problem that asks for understanding, not computation.*

Answering it for a given picture. *To say yes*, write down the cycle and check it, vertex by vertex: in Figure 14 the seating

1, 4, 7, 2, 9, 5, 3, 8, 6, 1

uses only edges of the graph and lists each of the nine students exactly once before returning to 1, so **yes**, such a seating exists. (On the sheet's own picture the right-hand copy of the graph has such a cycle highlighted, so the answer there is also *yes*; trace it and verify it edge by edge the same way.)

Verification is the mark-earning step: nine edges, nine distinct students.

What a "no" would look like. There is no quick test, so a negative answer must name an obstruction (Section 1.5). The three standard ones:

- *A student with only one friend.* A seating gives everyone two table neighbours, so a vertex of degree 1 makes it impossible.
- *A cut vertex.* If removing one student c splits the friendship graph into two or more pieces, there is no seating: deleting one vertex from a cycle leaves a path, which is still connected.
- *An imbalance.* If the students split into two groups with all friendships running between the groups (a bipartite graph), the seating must alternate between the groups, so the groups must have the same size. Seven students split $3 + 4$: impossible.

Each of these is a complete proof in two lines; pick whichever the picture supplies.

A sufficient condition worth trying first. By **Dirac's theorem**, if there are $k \geq 3$ students and every one of them has at least $k/2$ friends, a seating always exists. In Figure 14, $k = 9$ and the smallest degree is $3 < 4.5$, so Dirac does not apply, and we had to exhibit the cycle. That is typical.

Dirac is cheap to check; when it applies you are done without drawing anything.

How to search for a Hamilton cycle by hand

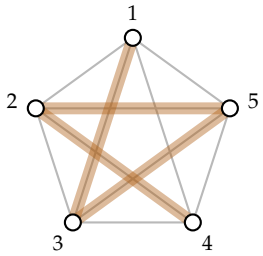
1. Forced edges first. Every vertex of degree 2 has *both* of its edges on the cycle, so mark them. **2. Propagate.** Once two edges at a vertex are marked, all its other edges can be deleted. **3. No short cycles.** Never close a cycle before all vertices are used. **4. Backtrack** when a vertex is left with fewer than two usable edges. On a graph of this size that is enough; for large graphs no good method is known, which is the point of the warning in Section 1.5.

2.10 P9: Hamilton paths and cycles in K_n

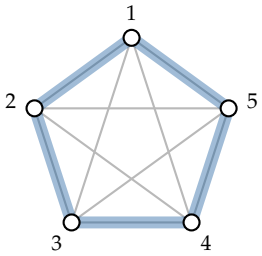
Problem P9

For which n does K_n contain a Hamilton path? A Hamilton cycle? Explain.

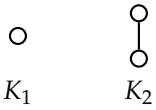
How to think about it. In K_n every pair of vertices is joined, so *every* ordering of the vertices is a path, and the only thing that can go wrong is the definition of a cycle: a cycle in a simple graph needs at least 3 vertices. So the answer is "always" for paths and "$n \geq 3$" for cycles, and the explanation is about being careful with $n = 1$ and $n = 2$.



Hamilton **path** $1, 3, 5, 2, 4$



Hamilton **cycle** $1, 2, 3, 4, 5, 1$



K_1 K_2

Hamilton path: yes (a single vertex; a single edge).
Hamilton cycle: **no** cycle at all.

Figure 15. K_5 with one Hamilton path and one Hamilton cycle highlighted, and the two exceptional cases K_1 and K_2.

Model solution to P9

Solution (what you write)

Commentary (what it does)

Set-up. Let K_n have vertices $v_1, v_2, \ldots, v_n$. By definition every two *distinct* vertices are joined by exactly one edge, and there are no loops.

The "no loops" part is what rules out $n = 1, 2$ for cycles.

Hamilton path: every $n \geq 1$. Take *any* ordering of the vertices, say $v_1, v_2, \ldots, v_n$. Consecutive vertices are distinct, hence adjacent, so this is a walk; and no vertex repeats, so it is a path; and it contains every vertex, so it is a Hamilton path. (For $n = 1$ it is the single vertex v_1, a path of length 0.) In fact *every* one of the $n!$ orderings works, giving $n!/2$ distinct Hamilton paths once you identify a path with its reverse.

"Every ordering works" is the whole proof; say why each consecutive pair is an edge.

Hamilton cycle: exactly $n \geq 3$. *Existence for $n \geq 3$:* the sequence

$$v_1,\ v_2,\ \ldots,\ v_{n-1},\ v_n,\ v_1$$

returns to its start, uses each vertex once in between, and every consecutive pair is an edge of K_n (including the closing pair $\{v_n, v_1\}$, since $v_n \neq v_1$ when $n \geq 2$). It has length $n \geq 3$, so it is a genuine cycle. Hence K_n is Hamiltonian for all $n \geq 3$.

Check the closing *edge separately; that is the one people forget.*

Non-existence for $n \leq 2$: K_1 has no edges and K_2 has a single edge, and a cycle in a simple graph has length at least 3 (a "cycle" of length 1 would need a loop, and one of length 2 would need the edge v_1v_2 twice, which is not allowed). So K_1 and K_2 contain no cycle whatsoever, Hamilton or otherwise.

This is a definition check, not a construction; state the definition you are using.

Answer.

K_n has a Hamilton path for every $n \geq 1$	K_n has a Hamilton cycle exactly for $n \geq 3$

Cross-check with the sufficient conditions. For $n \geq 3$ every vertex of K_n has degree $n - 1 \geq n/2$ as soon as $n \geq 2$, so **Dirac's theorem** also gives the cycle immediately. And *count*: K_n has $\frac{(n-1)!}{2}$ distinct Hamilton cycles for $n \geq 3$ (fix the starting vertex, order the other $n - 1$, and identify each cycle with its reverse). For $n = 5$ that is $4!/2 = 12$.

Contrast with K_n Eulerian only for odd n (Tutorial 7, Problem 7): Hamilton and Euler really are different questions.

Hamilton and Euler pull in different directions

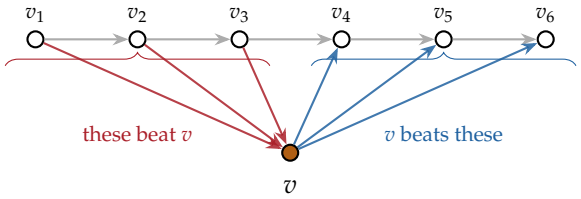
K_4 is Hamiltonian but has no Eulerian trail (four odd vertices). $K_{1,3}$ has neither. C_n has both. K_5 has both. There is no implication either way, so never argue "it is Eulerian, therefore ..." about Hamilton cycles.

2.11 P10: Rédei's theorem, by induction

Problem P10

Rédei's theorem: every tournament has a directed Hamilton path. Give an inductive proof.

How to think about it. Induct on the number of players. Pull one player v out of the tournament; what is left is a smaller tournament, so by the hypothesis the other players can be lined up in a chain $v_1 \to v_2 \to \cdots \to v_{n-1}$. Now you must slot v back in. Walk along the chain and ask, for each i, which way the arrow between v_i and v points. At the very front, either v beats v_1 (then v goes first and you are done), or v_1 beats v. At the very back, either v_{n-1} beats v (then v goes last) or v beats v_{n-1}. If neither end works, the answers must **switch** somewhere in the middle, from "$v_i \to v$" to "$v \to v_{i+1}$", and that switch is exactly a gap you can drop v into. There is no third possibility, because in a tournament every pair has an arrow one way or the other.



The colour changes between v_3 and v_4:
$v_3 \to v$ and $v \to v_4$.
New path: $v_1, v_2, v_3,\ v,\ v_4, v_5, v_6$.

Figure 16. The inductive step of Rédei's theorem. Each old vertex is red (it beats v) or blue (v beats it); in a tournament there is no third colour. If v_1 is blue, put v at the front; if v_{n-1} is red, put v at the back; otherwise the colours switch somewhere, and v drops into that gap.

Model solution to P10

Solution (what you write)	**Commentary (what it does)**
Statement. For every $n \geq 1$: every tournament T on n vertices contains a directed Hamilton path, that is, an ordering $w_1, w_2, \ldots, w_n$ of *all* its vertices with $w_i \to w_{i+1}$ an arc for $i = 1, \ldots, n-1$.	*Say "ordering of all its vertices": that is what Hamilton means here.*
Base case ($n = 1$). A single vertex w_1 is a directed path of length 0 containing every vertex. ✓ (The case $n = 2$ is also immediate: the single arc is the path.)	
Inductive hypothesis. Let $n \geq 2$ and assume that *every* tournament on $n-1$ vertices has a directed Hamilton path.	*Ordinary induction is enough here; strong induction is not needed.*

Inductive step. Let T be an **arbitrary** tournament on n vertices and pick any vertex v. Deleting v and all arcs at v leaves a digraph $T - v$ on $n - 1$ vertices in which every pair of distinct vertices still has exactly one arc between them, so $T - v$ is again a tournament. By the hypothesis it has a directed Hamilton path

$$P: \quad v_1 \to v_2 \to \cdots \to v_{n-1}.$$

Check that the smaller object still satisfies the hypothesis, as always in a graph induction.

Case 1: $v \to v_1$. Then $v, v_1, v_2, \ldots, v_{n-1}$ is a directed Hamilton path of T. ✓

Front insertion.

Case 2: $v_{n-1} \to v$. Then $v_1, \ldots, v_{n-1}, v$ is a directed Hamilton path of T. ✓

Back insertion.

Case 3: neither. Since T is a tournament, exactly one arc joins v to each v_i. Failing Case 1 means $v_1 \to v$; failing Case 2 means $v \to v_{n-1}$. So the set $S = \{\, i : v_i \to v \,\}$ contains 1 and does not contain $n - 1$; let i be its **largest** element, so $1 \le i \le n - 2$. By maximality $i + 1 \notin S$, and since exactly one arc joins v and v_{i+1}, this means $v \to v_{i+1}$. Therefore

"Take the largest i with $v_i \to v$" is the standard way to find the switch; it exists because $1 \in S$ and S is finite.

$$v_1,\ \ldots,\ v_i,\ v,\ v_{i+1},\ \ldots,\ v_{n-1}$$

is a directed path: the steps inside P are unchanged, and the two new steps are $v_i \to v$ (as $i \in S$) and $v \to v_{i+1}$. It lists every vertex of T exactly once. ✓

Conclusion. In all three cases T has a directed Hamilton path. By induction, every tournament has one. ■

Figure 16.

Another way to see it: take a longest path

Here is a proof with no induction at all. Let $P : v_1 \to \cdots \to v_k$ be a **longest** directed path in T and suppose, for a contradiction, that some vertex v is missing from it. Run the Case 1/2/3 analysis above: in every case v can be inserted, producing a directed path with $k + 1$ vertices, which contradicts the maximality of P. So no vertex is missing and P is a Hamilton path. ■

This is the same argument with "longest" doing the work that "induction" did, and it is worth knowing both: the longest-object trick is how many graph theory proofs avoid an explicit induction.

What Rédei's theorem does *not* give you

(i) Not a ranking. The path only says each player beats the *next* one. In the 3-cycle $a \to b \to c \to a$ the path a, b, c exists although c beats a: "beats" is not transitive.
(ii) Not a cycle. If one player beats everybody, no directed cycle can return to them. A directed Hamilton *cycle* needs the tournament to be **strongly connected**, which is Camion's theorem.
(iii) Not uniqueness. A tournament can have many Hamilton paths; the transitive tournament is the one with exactly one.

3 The additional problems, worked in full

These five problems come from the *Additional* sheet handed out with the tutorial. They use the same tools, but each one has a definite trick, and the trick is worth naming before the proof starts.

3.1 A1: a vertex of degree k in a tree forces k pendant vertices

Problem A1

Let T be a tree. If v is a vertex with degree k, prove that there are at least k pendant vertices in T.

How to think about it. Delete v. Because T is a tree, the pieces that fall off cannot be joined to one another, so there are exactly k of them, one per neighbour (Section 1.6). Each piece is itself a tree, and a tree always has a pendant vertex to spare. Collect one from each piece and you have k of them, all different because they live in different pieces. The only care needed is to check that a pendant vertex *of a piece* is still pendant *in* T, which is where the "to spare" matters.

Model solution to A1

Solution (what you write)	Commentary (what it does)
Statement. Let T be a tree and let $v \in V(T)$ with $\deg_T(v) = k \geq 1$. Then T has at least k pendant vertices (vertices of degree 1).	*The hypothesis $k \geq 1$ loses nothing: $\deg_T(v) = 0$ in a tree forces $T = K_1$, and "at least 0 pendant vertices" is trivially true there.*
Step 1: the pieces. Let $u_1, \ldots, u_k$ be the neighbours of v. By Section 1.6 the graph $T - v$ has exactly k connected components $T_1, \ldots, T_k$, with $u_i \in T_i$, and each T_i is a tree.	*This is where "T is a tree" is used: no second route between the pieces.*
Step 2: one pendant vertex of T from each piece. Fix i and consider T_i. • *If T_i has only one vertex,* that vertex is u_i. In T its only neighbour is v, so $\deg_T(u_i) = 1$ and u_i is a pendant vertex of T. Put $p_i = u_i$. • *If T_i has at least two vertices,* then T_i is a tree on ≥ 2 vertices, so it has at least **two** pendant vertices (Section 1.6). At most one of them is u_i, so we may choose a pendant vertex p_i of T_i with $p_i \neq u_i$. Then p_i is not adjacent to v in T, because u_i is the only vertex of T_i adjacent to v; hence $\deg_T(p_i) = \deg_{T_i}(p_i) = 1$, and p_i is a pendant vertex of T.	*"At least two, and at most one of them is u_i" is exactly why the piece has one* to spare.
Step 3: they are distinct. For $i \neq j$ the vertices p_i and p_j lie in different components of $T - v$, so $p_i \neq p_j$. Hence $p_1, \ldots, p_k$ are k distinct pendant vertices of T. ■	*Distinctness is a real step, not a formality; say where it comes from.*
Sharpness. The star $K_{1,k}$ has a vertex of degree k and exactly k pendant vertices, so "at least k" cannot be improved.	
Corollary. Every tree T on $n \geq 2$ vertices has at least $\Delta(T)$ pendant vertices, since $\Delta(T)$ is the degree of some vertex. This is how the result is usually quoted.	*Applying the statement at a maximum-degree vertex gives the strongest version for free.*

Another way to see it: induction on n

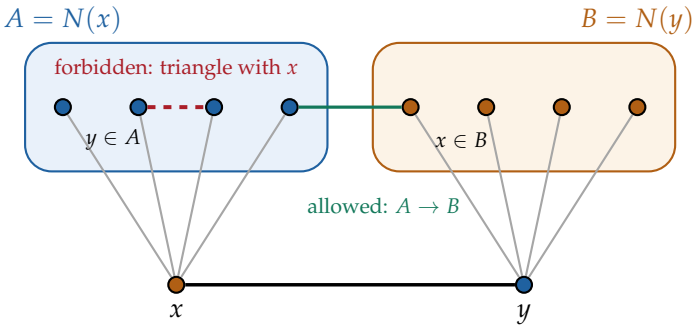
The same result can be proved by induction on the number of vertices, removing one pendant vertex at a time, but the bookkeeping is fussier: when you delete a pendant vertex p from T you may lower the degree of its neighbour, so the maximum degree can drop from Δ to $\Delta - 1$ and you have to split into two cases ("Δ survived" and "Δ dropped"), and in the second case you must check that the pendant vertices you inherit from the smaller tree are still pendant in T. The component argument above avoids all of that, which is why it is the one to write. If your lecturer presents the induction, make sure both cases are written out in full.

3.2 A2: a triangle-free degree condition forces bipartiteness

Problem A2

If there are two adjacent vertices x, y in a triangle-free graph G of order n such that $\deg(x) + \deg(y) \geq n$, prove that G is bipartite.

How to think about it. Two facts about a triangle-free graph with $xy \in E$: x and y have *no common neighbour* (that would close a triangle), and the neighbourhood of a vertex is an *independent set* (an edge inside it would close a triangle). The first fact says $N(x)$ and $N(y)$ are disjoint, so $\deg(x) + \deg(y) = |N(x) \cup N(y)| \leq n$. Together with the hypothesis $\geq n$ this forces equality, so $N(x)$ and $N(y)$ *partition* the whole vertex set. The second fact says both of those parts are independent. That is a bipartition.



$|A| + |B| = \deg x + \deg y \geq n$ and $A \cap B = \emptyset$, $A \cup B \subseteq V$, so $|A \cup B| = n$: A and B partition V, and both are independent.

Figure 17. The structure forced by the hypothesis. Note $y \in A$ and $x \in B$, so the edge xy itself runs between the two parts, as it must.

Model solution to A2

Solution (what you write)	Commentary (what it does)
Statement. Let G be a triangle-free graph on n vertices and let $x \sim y$ be adjacent vertices with $\deg(x) + \deg(y) \geq n$. Then G is bipartite.	*"Triangle-free" means G contains no cycle of length 3.*
Step 1: $N(x)$ and $N(y)$ are disjoint. Suppose $z \in N(x) \cap N(y)$. Then $x \sim z$, $y \sim z$ and $x \sim y$, so x, y, z is a triangle, contradicting the hypothesis. Hence $N(x) \cap N(y) = \emptyset$.	*The first of the two triangle-free rules.*

Step 2: they cover everything. Write $A = N(x)$ and $B = N(y)$. By Step 1,

$$|A \cup B| = |A| + |B| = \deg(x) + \deg(y) \;\geq\; n = |V|,$$

and $A \cup B \subseteq V$ gives $|A \cup B| \leq n$. So $|A \cup B| = n$ and $A \cup B = V$. Together with $A \cap B = \emptyset$, the pair (A, B) is a **partition** of V.

Squeeze: $\geq n$ and $\leq n$ force equality. This is the only place the hypothesis is used.

Note. $y \in A$ (since $y \sim x$) and $x \in B$ (since $x \sim y$), so both x and y are placed, and $x \notin A$, $y \notin B$ because G has no loops.

Worth one line: it shows the partition really does cover x and y.

Step 3: both parts are independent. Suppose $u, w \in A$ with $u \sim w$. Then $x \sim u$, $x \sim w$ and $u \sim w$, so x, u, w is a triangle: contradiction. Hence no edge has both ends in A. The same argument with y shows no edge has both ends in B.

The second triangle-free rule: a neighbourhood is independent.

Conclusion. $V = A \cup B$ with $A \cap B = \emptyset$, and every edge of G has one end in A and one in B (it cannot have both ends in the same part by Step 3). Therefore G is bipartite with parts A and B. ■

Two remarks. (i) The hypothesis also forces G to be **connected**, since every vertex is adjacent to x or to y. (ii) The bound is sharp: C_5 is triangle-free with $n = 5$ and every adjacent pair has $\deg(x) + \deg(y) = 4 = n - 1$, and C_5 is not bipartite. So "$\geq n$" cannot be weakened to "$\geq n - 1$".

Always look for the example that shows a hypothesis cannot be relaxed.

3.3 A3: every graph has a balanced orientation

Problem A3

In every graph G one can put a direction on the edges so that for all $v \in V(G)$, $\left|\deg^-(v) - \deg^+(v)\right| \leq 1$.

How to think about it. If a graph has an Eulerian circuit, orienting every edge in the direction you walk it gives $\deg^-(v) = \deg^+(v)$ at every vertex, because each visit uses one edge in and one edge out. So the problem is only about graphs that are not Eulerian, and the obstruction is odd degrees. Tutorial 7 told us there is an *even* number of odd-degree vertices, so they can be paired up. The cleanest way to pair them is to add one helper vertex u joined to all of them; then everything is even, we orient along Eulerian circuits, and finally we delete u again. Deleting u removes exactly one edge at each odd vertex, which is what pushes its balance from 0 to ± 1 and no further.

Model solution to A3

Solution (what you write)	Commentary (what it does)
Statement. Every graph G has an orientation in which $\left|\deg^-(v) - \deg^+(v)\right| \leq 1$ for every vertex v.	*"Orientation": one arrowhead per edge, no edge left undirected.*

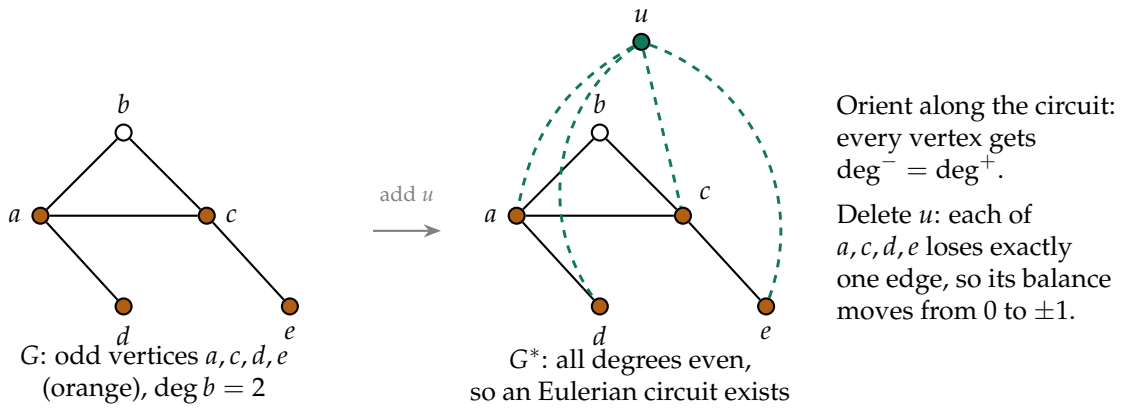


Figure 18. The helper vertex u is joined to every odd-degree vertex. There is an even number of them, so $\deg(u)$ is even too and the enlarged graph has an Eulerian circuit in each of its components.

Step 1: make all degrees even. By the Handshaking Theorem the number of odd-degree vertices of G is **even**; call them $z_1, \dots, z_{2m}$. Form G^* by adding one new vertex u and the $2m$ edges $uz_1, \dots, uz_{2m}$. In G^* every z_j has gained 1 and is now even, every other vertex of G is unchanged and even, and $\deg(u) = 2m$ is even. So **every** vertex of G^* has even degree.

Tutorial 7, corollary of the Handshaking Theorem. Without "even" there would be no way to pair them.

Step 2: orient along Eulerian circuits. G^* need not be connected, so work in each connected component separately. Each component is connected with all degrees even, so by Euler's theorem it has an Eulerian circuit (a component with no edges has nothing to orient). Walk each circuit once and orient every edge in the direction of travel.

"Component by component" is the step most solutions skip; Euler's theorem needs connectedness.

Step 3: the circuit balances every vertex. Fix a vertex w of a component and follow its Eulerian circuit. Every time the walk passes through w it arrives on one edge and leaves on another; since the circuit is closed, the very first departure from the starting vertex pairs with the very last arrival. So the edges at w are used in *in–out pairs*, and every edge at w is used exactly once. Hence in this orientation

$$\deg^-(w) = \deg^+(w) \qquad \text{for every vertex } w \text{ of } G^*.$$

This is the content of the proof; state it as an equality, not as "roughly balanced".

Step 4: delete the helper. Restrict the orientation to the edges of G, that is, delete u and its $2m$ arcs. Take any $v \in V(G)$.

- If $\deg_G(v)$ is even, no arc at v was deleted, so $\deg^-(v) = \deg^+(v)$ and the difference is 0.
- If $\deg_G(v)$ is odd, then $v = z_j$ for some j and exactly **one** arc at v was deleted, namely the one joining v to u. That drops either $\deg^-(v)$ or $\deg^+(v)$ by 1, so the difference becomes ± 1.

In both cases $\left|\deg^-(v) - \deg^+(v)\right| \le 1$. ■

Exactly one arc per odd vertex: that is why the bound is 1 and not more.

Sharpness. If $\deg(v)$ is odd then $\deg^-(v) + \deg^+(v)$ is odd, so the two can never be equal and the difference is at least 1. The orientation above is therefore optimal at every vertex.

The bound is not just achievable, it is forced.

Do not forget the components

"All degrees are even, so G^* has an Eulerian circuit" is false if G^* is disconnected. Two disjoint triangles have all degrees 2 and no Eulerian circuit. Handle each component on its own; this costs one sentence and is a standard place to lose a mark.

3.4 A4: a bipartite graph on n vertices has at most $n^2/4$ edges

Problem A4

Prove that the number of edges in every bipartite graph G on n vertices is at most $\dfrac{n^2}{4}$.

How to think about it. A bipartite graph can only use cross pairs, and there are rs of them when the parts have sizes r and s. So the whole problem reduces to a one-variable maximisation: how big can rs be when $r + s = n$? The product of two numbers with a fixed sum is largest when they are as equal as possible, which is the AM–GM inequality in disguise, and it is also just the vertex of a downward parabola.



Figure 19. With n vertices fixed, the number of available cross pairs $r(n-r)$ is a downward parabola in r, peaking at $r = n/2$. Splitting 6 vertices as $3+3$ beats $2+4$.

Model solution to A4

Solution (what you write)	Commentary (what it does)
Statement. If G is a bipartite graph on n vertices then $\lvert E(G)\rvert \le \dfrac{n^2}{4}$.	*G is simple; multiple edges would make the statement false.*
Step 1: bound by the cross pairs. Let (V_1, V_2) be a bipartition of G with $\lvert V_1\rvert = r$ and $\lvert V_2\rvert = s$, so $r + s = n$ with $r, s \ge 0$. Every edge of G has one end in V_1 and one in V_2, and distinct edges give distinct pairs since G is simple. There are exactly rs such pairs, so $$\lvert E(G)\rvert \;\le\; rs \;=\; r(n-r).$$	*The maximum rs is attained by $K_{r,s}$ itself, so nothing is lost here.*

Step 2: maximise $f(r) = r(n-r)$. Completing the square,

$$r(n-r) \;=\; -\Big(r-\frac{n}{2}\Big)^2 + \frac{n^2}{4} \;\le\; \frac{n^2}{4},$$

with equality exactly when $r = n/2$. Hence $|E(G)| \le r(n-r) \le \dfrac{n^2}{4}$. ■

Completing the square is cleaner than calculus and needs no derivative.

The integer version. r must be a whole number, so the true maximum is

$$\max_{0\le r\le n} r(n-r) \;=\; \Big\lfloor \frac{n}{2} \Big\rfloor \Big\lceil \frac{n}{2} \Big\rceil \;=\; \Big\lfloor \frac{n^2}{4} \Big\rfloor.$$

For odd n the bound $n^2/4$ is never attained, and the sharp bound is $\lfloor n^2/4 \rfloor$.

n even: $r = s = n/2$ gives $\frac{n^2}{4}$ exactly, attained by $K_{n/2,n/2}$.
n odd: the best split is $\frac{n-1}{2}$ and $\frac{n+1}{2}$, giving $\frac{n-1}{2}\cdot\frac{n+1}{2} = \frac{n^2-1}{4} < \frac{n^2}{4}$, attained by $K_{(n-1)/2,\,(n+1)/2}$.

Check. $n = 6$: $\lfloor 36/4 \rfloor = 9 = |E(K_{3,3})|$. ✓ $n = 7$: $\lfloor 49/4 \rfloor = 12 = |E(K_{3,4})|$, and $49/4 = 12.25$. ✓

Where this comes from. The same number $\lfloor n^2/4 \rfloor$ is the answer to **Mantel's theorem**: a *triangle-free* graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges, with $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ the unique extremal graph. Bipartite graphs are triangle-free, so A4 is the easy half of Mantel's theorem, and the extremal example is the same.

Tutorial 7, Problem 10 used Mantel's theorem; this is where its bound comes from.

3.5 A5: every graph sits inside a regular graph

Problem A5

(a) Consider the graph consisting of a triangle with one extra vertex attached by a single edge. Show that this graph is a subgraph of a 3-regular graph.
(b) Prove that every graph G with maximum degree Δ is a subgraph of a Δ-regular graph.
(c) Prove that every *bipartite* graph G with maximum degree Δ is a subgraph of a bipartite Δ-regular graph.

How to think about it. You cannot simply add edges inside G: a vertex might already be joined to everything it is allowed to be joined to. So you must add *new vertices*, and the cheapest way is to take a second copy G' of G and join each deficient vertex to its own twin. Every vertex that was short of Δ gains exactly one, so the **minimum degree goes up by one** and the maximum stays at Δ. Repeat: after $\Delta - \delta$ rounds everything has reached Δ. The bipartite version is the same construction, and the only new thing to check is that the result is still bipartite, which it is if you cross the two bipartitions over.

Model solution to A5(b), the general construction

Solution (what you write)	Commentary (what it does)

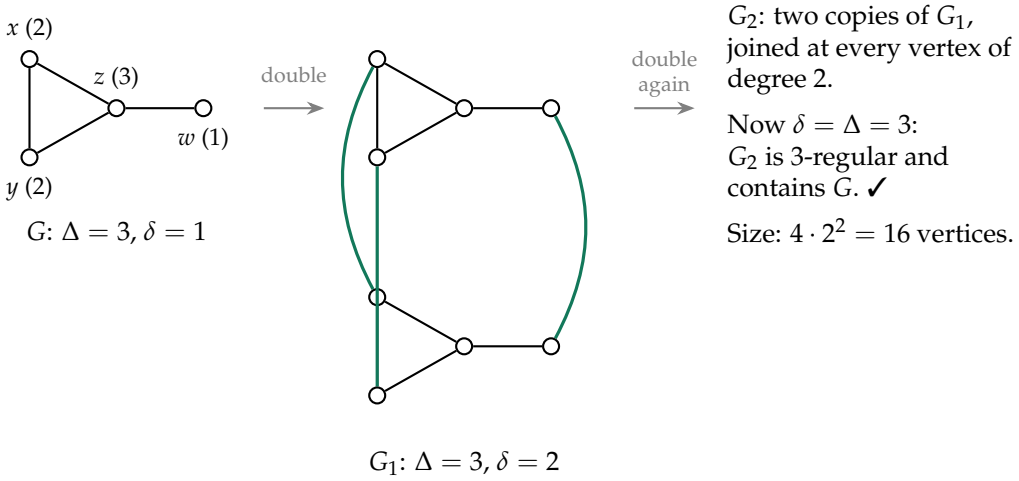


Figure 20. The doubling construction for A5(a). Each round joins every deficient vertex to its twin, so the minimum degree climbs by exactly 1 *and the maximum never moves.*

Statement. Let G be a graph with maximum degree Δ. Then G is a subgraph of some Δ-regular graph.

Prove (b) first; (a) is then an instance and (c) a refinement.

The doubling operation. Given a graph H with $\Delta(H) \le \Delta$, define $D(H)$ as follows. Take two disjoint copies of H; write u and u' for the two copies of each vertex u of H. Add the edge $\{u, u'\}$ for **every** vertex u with $\deg_H(u) < \Delta$, and no other edges.

One sentence; everything below is a computation about it.

What doubling does to the degrees. Let u be a vertex of H. In $D(H)$ its copy has degree

The two bullet points are the whole proof: max fixed, min +1.

$$\deg_{D(H)}(u) = \begin{cases} \deg_H(u) + 1, & \text{if } \deg_H(u) < \Delta, \\ \deg_H(u) = \Delta, & \text{if } \deg_H(u) = \Delta, \end{cases}$$

and the same for u'. Consequently

$$\Delta\big(D(H)\big) = \Delta \qquad \text{and} \qquad \delta\big(D(H)\big) = \min\big(\delta(H) + 1,\ \Delta\big).$$

A vertex below Δ gains exactly 1 and so cannot overshoot; a vertex already at Δ gains nothing.

Iterate. Put $H_0 = G$ and $H_{t+1} = D(H_t)$. By the formula above, $\Delta(H_t) = \Delta$ for all t and $\delta(H_t) = \min(\delta(G) + t,\ \Delta)$. Taking $t = \Delta - \delta(G)$ gives $\delta(H_t) = \Delta = \Delta(H_t)$, so

Finitely many rounds, and you can say exactly how many.

$$H_{\Delta - \delta(G)} \text{ is } \Delta\text{-regular.}$$

G **really is a subgraph.** Each copy of H_t inside H_{t+1} keeps all of its own edges, so by induction the "first copy" of G survives, with all of its edges, inside $H_{\Delta-\delta(G)}$. Hence G is a subgraph of a Δ-regular graph. ■

Do not skip this: the question asks for a subgraph*, not just for existence of a regular graph.*

Size. Each round doubles the vertex count, so the final graph has $|V(G)| \cdot 2^{\Delta-\delta(G)}$ vertices. This is wasteful but the question asks only for existence; much smaller constructions exist.

Honest accounting; mention it rather than pretending the construction is efficient.

Parity caveat. A Δ-regular graph on N vertices needs $N\Delta$ even. This is exactly what can block adding edges inside G (see the warning below), and it is exactly what the construction sidesteps: after the first doubling N is even, so the condition is free. If no doubling is needed then G was already Δ-regular and $N\Delta$ was already even.

A good check that the construction could not have been "obviously impossible".

Model solution to A5(a) and A5(c)

Solution (what you write) | **Commentary (what it does)**

(a) The concrete graph. G is the triangle x, y, z with a fourth vertex w joined only to z. Its degrees are $\deg x = \deg y = 2$, $\deg z = 3$, $\deg w = 1$, so $\Delta = 3$ and $\delta = 1$. By (b) we need $\Delta - \delta = 3 - 1 = 2$ rounds of doubling.

Compute Δ and δ first; they tell you the number of rounds.

Round 1. Take two copies of G and join x–x', y–y' and w–w' (the vertices of degree < 3); do *not* join z–z', since z already has degree 3. The result G_1 has 8 vertices with degrees: x, x', y, y' and z, z' at 3, and w, w' at 2. So $\Delta = 3$, $\delta = 2$.

Figure 20.

Round 2. Take two copies of G_1 and join only the four copies of w to their twins. Now every one of the 16 vertices has degree 3: the graph G_2 is 3-regular, and the original G sits inside it as one of its four copies. ■

$4 \cdot 2^2 = 16$ *vertices, as predicted.*

(c) The bipartite case. Let G be bipartite with parts A and B and maximum degree Δ. Run exactly the same doubling, but describe the bipartition of the result. Write A', B' for the copies of A, B in the second copy G'. Put

$$\widehat{A} = A \cup B', \qquad \widehat{B} = B \cup A'.$$

The crossing-over $A \cup B'$ / $B \cup A'$ is the only idea in part (c).

These sets partition the vertex set of $D(G)$. Every edge inside G joins A to B, hence $\widehat{A}$ to $\widehat{B}$. Every edge inside G' joins A' to B', hence again $\widehat{B}$ to $\widehat{A}$. And every new twin edge $\{u, u'\}$ has $u \in A, u' \in A'$ (so $\widehat{A}$ to $\widehat{B}$) or $u \in B$, $u' \in B'$ (so $\widehat{B}$ to $\widehat{A}$). In all three cases the edge crosses, so $D(G)$ is **bipartite**.

Finish. Doubling therefore preserves bipartiteness as well as Δ, and raises δ by 1. After $\Delta - \delta(G)$ rounds the result is a **bipartite Δ-regular** graph containing G. ■

Why the bipartite version is believable. A bipartite Δ-regular graph must have parts of *equal* size, since counting edges from each side gives $\Delta|A| = |E| = \Delta|B|$. The construction takes care of this automatically: in $D(G)$ the parts are $A \cup B'$ and $B \cup A'$, both of size $|A| + |B|$.

A good sanity check that the crossing-over was the right choice.

Why you cannot just add edges inside G

Take C_5 plus one chord: $v = 5$, $\Delta = 3$, and three vertices still have degree 2. No set of extra edges on those five vertices can make it 3-regular, because $5 \cdot 3 = 15$ is odd and $\sum$deg must be even. So no 3-regular graph on 5 vertices exists at all, and any proof must create new vertices. Equally, you cannot add a single new vertex joined to all deficient vertices: its own degree would usually be wrong. Doubling is the standard fix because it changes every deficient vertex by exactly 1 and nothing else.

4 Mistakes, checklist and exercises

4.1 Common mistakes (and how to avoid them)

Mistakes that cost marks

1. **Using a bound that holds.** $e \leq 3v - 6$ is necessary for planarity, never sufficient. A satisfied bound proves nothing (P2, P4).
2. **Forgetting the infinite region** when counting regions, or forgetting that Euler's formula needs the graph to be **connected** ($v - e + r = 1 + k$ for k components).
3. **Counting a pendant edge once** in a region degree. An edge with the same region on both sides is counted **twice** in $\sum_R \deg(R) = 2e$.
4. **Circular use of Kuratowski.** "K_5 is non-planar by Kuratowski" is empty. Prove the two base cases with the edge bounds.
5. **Paths that are not internally disjoint** in a claimed subdivision of $K_{3,3}$, or the wrong number of branch vertices (there must be exactly six, each of degree 3 in the subgraph).
6. **Confusing length with vertex count.** A path of length n has $n + 1$ vertices (P5). Off-by-one here reverses the answer.
7. **Trail versus tour** in the bridge-builder problem. "Every bridge once" needs ≤ 2 odd vertices, so one new bridge; "and return home" needs 0, so two.
8. **Treating Hamilton like Euler.** There is no degree test for Hamilton cycles. "Every degree is even, so there is a Hamilton cycle" is nonsense.
9. **"I could not find one, so there is none"** for Hamilton cycles. You must name an obstruction.
10. **Forgetting the closing edge** of a Hamilton cycle, or the case $n \leq 2$ in P9.
11. **Forgetting components** when applying Euler's theorem to the enlarged graph in A3.
12. **Adding edges inside** G in A5, where there may be none left to add. New vertices are required.
13. **Using $n^2/4$ as if it were attainable for odd n** in A4. The sharp bound is $\lfloor n^2/4 \rfloor$.

Spot the error 1: planarity from a degree sequence

"*Solution.*" The degrees are $2, 2, 3, 4, 4, 5$, so $e = 10$. A planar graph on 6 vertices can have up to $3 \cdot 6 - 6 = 12$ edges, and $10 \leq 12$, so G is planar.
Where is the mistake? (Answer in Appendix A.)

Spot the error 2: counting regions

"*Solution.*" My drawing of the octahedron has 6 vertices and 12 edges. I count the bounded regions and find 7 of them, so $v - e + r = 6 - 12 + 7 = 1$, and Euler's formula must be wrong.
Where is the mistake? (Answer in Appendix A.)

Spot the error 3: Kuratowski on the Petersen graph

"*Solution.*" The Petersen graph is 3-regular with 10 vertices. Pick any six of them as branch vertices; since the graph is vertex-transitive, the nine required paths exist, so it contains a subdivision of $K_{3,3}$ and is non-planar.
Where is the gap? (Answer in Appendix A.)

Spot the error 4: the complement of $K_{m,n}$

"*Solution.*" $\overline{K_{m,n}} = K_m \cup K_n$. Its longest path has $m + n$ vertices, so it contains a path of length m and a path of length n for all $m, n \geq 1$.
Where is the mistake? (Answer in Appendix A.)

Spot the error 5: the round table

"*Solution.*" Every student in the picture has at least two friends and the graph is connected, so a seating exists.
Where is the gap? (Answer in Appendix A.)

Spot the error 6: the bridge builder

"*Solution.*" Königsberg has four odd vertices. Each new bridge makes one odd vertex even, so we need four new bridges.
Where is the mistake? (Answer in Appendix A.)

Spot the error 7: Rédei's theorem

"*Inductive step.*" Remove a vertex v; by the hypothesis the rest has a directed Hamilton path $v_1 \to \cdots \to v_{n-1}$. If $v \to v_1$ put v first, and otherwise put v last.
Where is the gap? (Answer in Appendix A.)

Spot the error 8: pendant vertices in a tree

"*Solution.*" Delete the vertex v of degree k. Each of the k components is a tree, and every tree has at least two pendant vertices, so T has at least $2k$ pendant vertices.
Where is the mistake? (Answer in Appendix A.)

4.2 Checklist before you hand in

Checklist

- ☐ I said whether the graph is simple or a multigraph, and whether it is connected.
- ☐ Every region count includes the **infinite region**, and $v - e + r = 2$ checks out.
- ☐ Where I used $e \leq 3v - 6$ or $e \leq 2v - 4$, I stated the hypotheses (simple, $v \geq 3$, triangle-free) and said whether the bound **failed** (proof) or **held** (no information).
- ☐ For a Kuratowski argument: exactly six (or five) branch vertices named, all nine (or ten) paths listed, interior vertices checked to be distinct.
- ☐ For a planarity claim, I produced an actual drawing, not just a satisfied inequality.
- ☐ "Length" means edges everywhere in my solution, and I converted to vertex counts where needed.
- ☐ For a Hamilton "yes" I wrote the cycle out and verified each consecutive pair; for a "no" I named a specific obstruction.
- ☐ For an Eulerian repair I said whether I was aiming at a trail or a tour, and justified the count with the parity argument.
- ☐ In every induction, the smaller object **still satisfies the hypothesis** (still a tree, still a tournament, still even degrees), and I started from an **arbitrary** larger object.
- ☐ Where Euler's theorem was applied, the graph was connected, or I worked component by component.
- ☐ My small cases ($n = 1, 2$; K_1, K_2; a vertex of degree 0 or 1) were checked.
- ☐ Every proof ends with a conclusion sentence and ■.

4.3 Exercises

Write each solution in full, using the templates. Answers and hints are in Appendix B, so this section works as self-study: try each one for ten minutes before you look.

E1. A connected planar graph has $v = 10$ and $r = 7$. How many edges does it have? Could it be bipartite?

E2. Is there a planar graph with 7 vertices in which every vertex has degree 5? Give one or prove there is none.

E3. The **dodecahedron** has 12 pentagonal faces, and three faces meet at every vertex. Use $\sum_R \deg(R) = 2e$ and Euler's formula to find v and e without drawing it.

E4. Show that $K_{3,3}$ minus one edge *is* planar, by drawing it. What does this say about the sharpness of the bound $e \leq 2v - 4$?

E5. Prove that every planar graph with $v \geq 3$ that contains a cycle, but no cycle of length less than 5, satisfies $e \leq \frac{5}{3}(v - 2)$, and use it to re-prove that the Petersen graph is not planar. (Where does the argument break if G has no cycle at all?)

E6. For which $m, n \geq 1$ is $\overline{K_{m,n}}$ connected? For which m, n is each of its two components Hamiltonian?

E7. For which m, n does $K_{m,n}$ have (a) a Hamilton cycle, (b) a Hamilton path? Prove both directions.

E8. Prove that Q_n has a Hamilton cycle for every $n \geq 2$. (Hint: induction; Q_{n+1} is two copies of Q_n joined by a perfect matching.)

E9. A connected graph has exactly $2k$ vertices of odd degree, with $k \geq 1$. Prove that its edge set can be partitioned into k trails, and no fewer.

E10. Let T be a tree with exactly two pendant vertices. Prove that T is a path.

E11. Prove that a graph is bipartite if and only if it has no cycle of odd length. (Hint for the harder direction: fix a vertex x in a connected graph and split the vertices by the parity of their distance

from x.)

E12. Give a tournament on 4 vertices with exactly one directed Hamilton path, and one with more than one. What is special about the first?

E13. Prove that a graph with a cut vertex has no Hamilton cycle. Then show that the converse fails: give a graph with no cut vertex (indeed 3-connected) and no Hamilton cycle.

E14. Let G be a simple graph with n vertices and more than $\lfloor n^2/4 \rfloor$ edges. Prove that G contains a triangle. (This is Mantel's theorem; use A2.)

E15. A graph G has degree sequence $3, 3, 3, 3, 3, 3$ (6 vertices). List all such graphs up to isomorphism and decide for each whether it is planar and whether it is Hamiltonian.

A Answers to the "spot the error" puzzles

Spot the error 1 (a bound that holds proves nothing). $e \leq 3v - 6$ is a *necessary* condition for planarity, not a sufficient one. Satisfying it does not make a graph planar: $K_{3,3}$ has $v = 6$, $e = 9 \leq 12$ and is not planar. The conclusion happens to be true for this degree sequence, but only because we can produce an actual drawing (P2, Figure 10); without a drawing or a Kuratowski argument, nothing has been proved.

Spot the error 2 (the infinite region). The count of 7 bounded regions is correct; what is missing is the **unbounded** region outside the whole drawing. With it, $r = 8$ and $6 - 12 + 8 = 2$. ✓ Euler's formula always counts the infinite region.

Spot the error 3 ("pick any six"). Vertex-transitivity says that the graph looks the same from every vertex; it says nothing about arbitrary *sets* of six vertices, and it certainly does not produce nine internally disjoint paths. You must exhibit the paths, and for that the right six vertices matter: in P4 we first deleted two edges precisely so that exactly six vertices kept degree 3. A "proof" that names no paths is not a proof.

Spot the error 4 (paths cannot cross components). $K_m \cup K_n$ is **disconnected**, and a path is a connected subgraph, so it lies inside K_m or inside K_n. Its longest path has $\max(m, n)$ vertices, not $m + n$, so its length is $\max(m, n) - 1$. A path of length n exists only when $m \geq n + 1$, and one of length m only when $n \geq m + 1$; when $m = n$ neither exists.

Spot the error 5 (no sufficient condition was used). Minimum degree 2 and connectedness are *necessary* for a Hamilton cycle, not sufficient. The bowtie (two triangles sharing a vertex) is connected with all degrees ≥ 2 and has no Hamilton cycle. To answer "yes" you must exhibit a cycle, or verify a genuine sufficient condition such as Dirac's ($\delta \geq v/2$) or Ore's.

Spot the error 6 (each bridge flips *two* parities). A new bridge has two ends, so it changes the degree of two vertices and flips both parities. One well-chosen bridge therefore makes two odd vertices even. Also, an Eulerian *trail* is allowed to keep two odd vertices; we do not need zero. Starting from 4 odd vertices, one bridge suffices, and the parity argument shows one is also necessary.

Spot the error 7 (the middle case is missing). If $v_1 \to v$ and $v \to v_{n-1}$, then neither "v first" nor "v last" is a directed path, and the argument stops. The missing case is the insertion in the middle: take the largest i with $v_i \to v$; then $i \leq n - 2$ and, because a tournament has an arc between every pair, $v \to v_{i+1}$, so v goes between v_i and v_{i+1} (P10, Case 3).

Spot the error 8 (the double count). A tree on *one* vertex has no pendant vertex of its own, and a component of $T - v$ may well be a single vertex, so "every tree has at least two pendant vertices" does not apply to it. Worse, even for a larger component T_i, one of its two pendant vertices may be u_i, which is *not* pendant in T because it is also joined to v. Each component is therefore guaranteed

to contribute exactly **one** pendant vertex of T, not two, and the correct bound is k, which is attained by the star $K_{1,k}$.

B Answers and hints for the exercises

E1. Euler's formula gives $e = v + r - 2 = 10 + 7 - 2 = \boxed{15}$. *Bipartite?* **Yes, it can be.** The triangle-free bound allows it, since $e = 15 \leq 2v - 4 = 16$ (equivalently $4r = 28 \leq 2e = 30$). And an example exists: take parts $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, \ldots, b_7\}$ with the 15 edges a_1b_j for all 7 values of j, a_2b_j for $j = 1, 2, 4, 6, 7$, and a_3b_j for $j = 1, 3, 5$; this is connected, bipartite and planar with $v = 10$, $e = 15$ and $r = 7$. (A graph with these numbers is *not forced* to be bipartite, of course: the point of the question is that the counts alone do not decide.)

E2. **No such graph exists at all**, planar or not: $\sum \deg = 7 \cdot 5 = 35$ is odd, contradicting the Handshaking Theorem. (Even if the parity worked, $e = 35/2$ would exceed $3v - 6 = 15$.)

E3. Each of the 12 regions is a pentagon, so $\sum_R \deg(R) = 12 \cdot 5 = 60 = 2e$, giving $\boxed{e = 30}$. Euler's formula then gives $v = 2 + e - r = 2 + 30 - 12 = \boxed{20}$. Check with the vertex condition: three faces meet at each vertex means the graph is 3-regular, and $\frac{20 \cdot 3}{2} = 30 = e$. ✓

E4. $K_{3,3}$ minus one edge has $v = 6$ and $e = 8$. Draw it as a hexagon $a_1b_1a_2b_2a_3b_3$ with the two remaining chords a_1b_2 and a_2b_3 placed one inside and one outside the hexagon: no crossings, so it **is** planar. Since $2v - 4 = 8 = e$, this shows the triangle-free bound is **sharp**: a bipartite planar graph can have exactly $2v - 4$ edges, and $K_{3,3}$ fails planarity by the smallest possible margin, one edge.

E5. Every cycle has length ≥ 5, so every region of an embedding has $\deg(R) \geq 5$, giving $5r \leq \sum_R \deg(R) = 2e$ and $r \leq \frac{2e}{5}$. Putting this into $2 = v - e + r$ gives $2 \leq v - e + \frac{2e}{5} = v - \frac{3e}{5}$, hence $e \leq \frac{5}{3}(v - 2)$. For the Petersen graph, $v = 10$ and girth 5, so planarity would force $e \leq \frac{5}{3} \cdot 8 = 13\frac{1}{3}$, that is $e \leq 13$. But $e = 15$. Contradiction, so it is not planar.

E6. $\overline{K_{m,n}} = K_m \cup K_n$ has exactly **two** components whenever $m, n \geq 1$, so it is **never** connected, and in particular never Hamiltonian as a whole. Its component K_t is Hamiltonian exactly when $t \geq 3$, so both components are Hamiltonian exactly when $\boxed{m \geq 3 \text{ and } n \geq 3}$.

E7. *(a) Hamilton cycle* $\iff$ $m = n \geq 2$. ($\Rightarrow$) $K_{m,n}$ is bipartite, and a cycle alternates between the parts, so a Hamilton cycle forces $m = n$; and a cycle has length at least 3, so $2n \geq 3$, i.e. $n \geq 2$. ($\Leftarrow$) For $m = n \geq 2$ the cycle $x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1$ works. *(b) Hamilton path* $\iff$ $|m - n| \leq 1$. ($\Rightarrow$) A path alternates, so the two parts differ by at most 1. ($\Leftarrow$) If $m = n$, take $x_1, y_1, \ldots, x_n, y_n$; if $m = n + 1$, take $x_1, y_1, x_2, y_2, \ldots, y_n, x_{n+1}$.

E8. Induction. *Base* $n = 2$: $Q_2 = C_4$ is a Hamilton cycle. *Step:* Q_{n+1} consists of two copies Q^0 (strings starting with 0) and Q^1 (starting with 1) of Q_n, plus the perfect matching joining each $0x$ to $1x$. By the hypothesis Q^0 has a Hamilton cycle C; let $0x$ and $0y$ be two *consecutive* vertices on it. Delete the edge $\{0x, 0y\}$ to get a Hamilton *path* of Q^0 from $0x$ to $0y$, take the mirror-image Hamilton path of Q^1 from $1y$ to $1x$, and join the two with the matching edges $\{0y, 1y\}$ and $\{1x, 0x\}$. The result is a Hamilton cycle of Q_{n+1}. (This is exactly the recursive construction of a *Gray code*.)

E9. *At most k trails.* Pair the $2k$ odd vertices arbitrarily and add k new edges joining the pairs (allowing multiple edges). Every degree is now even and the graph is still connected, so it has an Eulerian circuit. Deleting the k added edges from the circuit breaks it into exactly k trails, which use every original edge exactly once. *At least k trails.* In any partition of E into trails, every vertex of odd degree must be an endpoint of at least one trail (otherwise the trails would use its edges in in–out pairs, an even number). Each trail has at most two endpoints, so the number of trails is at

least $2k/2 = k$.

E10. Let T be a tree on n vertices with exactly two pendant vertices. Then

$$2(n-1) = 2|E(T)| = \sum_x \deg(x) \;\geq\; 1+1+2(n-2) = 2n-2 = 2(n-1),$$

since the two pendant vertices contribute 1 each and every other vertex has degree ≥ 2. Equality throughout forces every non-pendant vertex to have degree exactly 2. A connected graph in which all degrees are ≤ 2 is a path or a cycle, and a tree has no cycle, so T is a path.

E11. ($\Rightarrow$) In a bipartite graph every edge switches part, so a closed walk switches an even number of times and every cycle has even length. ($\Leftarrow$) It is enough to treat one connected component. Fix a vertex x and put $V_1 = \{\, u : d(x,u) \text{ is even} \,\}$, $V_2 = \{\, u : d(x,u) \text{ is odd} \,\}$, where d is the distance. If an edge uw had both ends in the same part, then the shortest paths $x \to u$ and $x \to w$ together with uw would produce a closed walk of odd length, and a closed walk of odd length always contains a cycle of odd length. That contradicts the hypothesis, so every edge crosses and the graph is bipartite.

E12. The **transitive** tournament on $\{1,2,3,4\}$, in which $i \to j$ exactly when $i < j$, has exactly one directed Hamilton path, namely $1,2,3,4$: it is a total order, so the chain is forced. A tournament with more: take the 4-cycle $a \to b \to c \to d \to a$ together with $a \to c$ and $d \to b$; both a,b,c,d and d,a,b,c are directed Hamilton paths (Figure 8). *What is special about the first:* it is transitive, and the transitive tournament is the *only* one on any number of vertices with a unique directed Hamilton path. Checking all $2^6 = 64$ tournaments on 4 vertices gives 1, 3 or 5 Hamilton paths and never an even number. That is a special case of a known strengthening of Rédei's theorem: the number of directed Hamilton paths in any tournament is always **odd**. The 64-case check illustrates it; it does not prove it.

E13. Suppose G has a cut vertex c, so $G - c$ has at least two components. If C were a Hamilton cycle of G, then $C - c$ would be a Hamilton *path* of $G - c$, so $G - c$ would be connected: contradiction. Hence no Hamilton cycle. *The converse fails:* $K_{3,4}$ is 3-connected (no cut vertex at all) and has no Hamilton cycle, by the bipartite imbalance $3 \neq 4$. The **Petersen graph** is another example, also 3-connected and not Hamiltonian, and there the proof is much harder: no small obstruction works.

E14. Let G be triangle-free on n vertices with e edges; we show $e \leq n^2/4$. If some edge xy has $\deg(x) + \deg(y) \geq n$, then by A2 the graph is bipartite, and by A4 $e \leq n^2/4$, as required. Otherwise every edge satisfies $\deg(x) + \deg(y) \leq n-1$. Summing over the edges and regrouping by vertices,

$$\sum_v \deg(v)^2 \;=\; \sum_{xy \in E} \big(\deg(x) + \deg(y)\big) \;\leq\; (n-1)\, e,$$

while the Cauchy–Schwarz inequality gives $\sum_v \deg(v)^2 \;\geq\; \frac{1}{n}\big(\sum_v \deg(v)\big)^2 = \frac{4e^2}{n}$. Combining, $\frac{4e^2}{n} \leq (n-1)e$. If $e = 0$ the conclusion is immediate; otherwise divide by e to get $e \leq \frac{n(n-1)}{4} < \frac{n^2}{4}$. Either way $e \leq n^2/4$; contrapositively, more than $\lfloor n^2/4 \rfloor$ edges forces a triangle. *(The identity* $\sum_{xy \in E}(\deg x + \deg y) = \sum_v \deg(v)^2$ *holds because each vertex v contributes* $\deg(v)$ *to the left side once for each of its* $\deg(v)$ *edges.)*

E15. There are exactly **two** 3-regular graphs on 6 vertices up to isomorphism.

- The **triangular prism** (two triangles joined by a perfect matching): **planar**, not bipartite (it has triangles), **Hamiltonian** (Tutorial 7, Figure 8).
- $K_{3,3}$: **not planar** (P9 of Tutorial 7), bipartite, **Hamiltonian** ($x_1, y_1, x_2, y_2, x_3, y_3, x_1$).

Both have $e = \frac{6 \cdot 3}{2} = 9$. Note that the prism satisfies $e = 9 \leq 3v - 6 = 12$ and so does $K_{3,3}$; only the triangle-free bound $2v - 4 = 8 < 9$ separates them, which is this tutorial's recurring lesson.