Tutorial 10: Complements and Planarity, Duals, Region Counting and Bipartite Structure

the edge-budget argument, dual graphs, counting with region degrees, degree-sum extremal problems,
odd cycles, properties inherited by subgraphs, joins and complements

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Use the **edge-budget argument**: if G and $\overline{G}$ both have a property that caps the number of edges, the two caps together must cover all $\binom{n}{2}$ pairs.
2. Build the **dual** G^d of a plane graph and translate between vertices, edges and regions of G and G^d.
3. Turn information about **region degrees** into bounds on v, e and r with Euler's formula, and round correctly at the end.
4. Solve **degree-sum extremal problems**: prove a bound with the Handshaking Theorem, then *attain* it with an explicit graph.
5. Prove a graph is **bipartite** by producing the two parts, and understand why bipartite graphs and odd cycles exclude each other.
6. Recognise properties that **pass to subgraphs** and prove it by "restricting the witness".
7. Compute complements of **joins** and use them to build graphs with prescribed complements.

Section 1 is the toolbox: each topic has the intuition, a simple example, the formal statement, a comprehensive example and a recap. Section 2 collects common mistakes and "spot the error" puzzles, and Section 3 has practice problems. Answers to everything are in the appendix, so the notes work for **self-study**. Tutorials 7 to 9 (Handshaking, Euler's formula $v - e + r = 2$, the bound $e \leq 3v - 6$, complements, bipartite graphs) are assumed.

Conventions used throughout

A **graph** is finite, undirected and **simple** (no loops, no multiple edges) unless stated otherwise; $v = |V|$, $e = |E|$, and r is the number of regions of a planar embedding, **including the infinite region**. $\delta(G)$ is the minimum degree. The **degree of a region**, $\deg(R)$, is the number of edge-sides met walking once around its boundary, so $\sum_R \deg(R) = 2e$ (Tutorial 8). Theorem numbers such as *Theorem 11.6* and *Corollary 11.3* refer to the course text (Grimaldi, Chapter 11).

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1 The toolbox

1.1 The edge-budget argument for a graph and its complement

The intuition

Paint every pair of vertices of K_n black (an edge of G) or white (an edge of $\overline{G}$). There are $\binom{n}{2}$ pairs and every one of them gets a colour. Now suppose some property, such as "no cycle" or "planar", allows a graph on n vertices to have **at most** $f(n)$ edges. If both the black graph and the white graph had the property, the two of them together could hold at most $2f(n)$ pairs. So as soon as

$$\binom{n}{2} > 2f(n),$$

there is not enough room: **at least one** of G, $\overline{G}$ must fail the property. Because $\binom{n}{2}$ grows like $n^2/2$ and the usual caps grow like a constant times n, this always happens for large n.

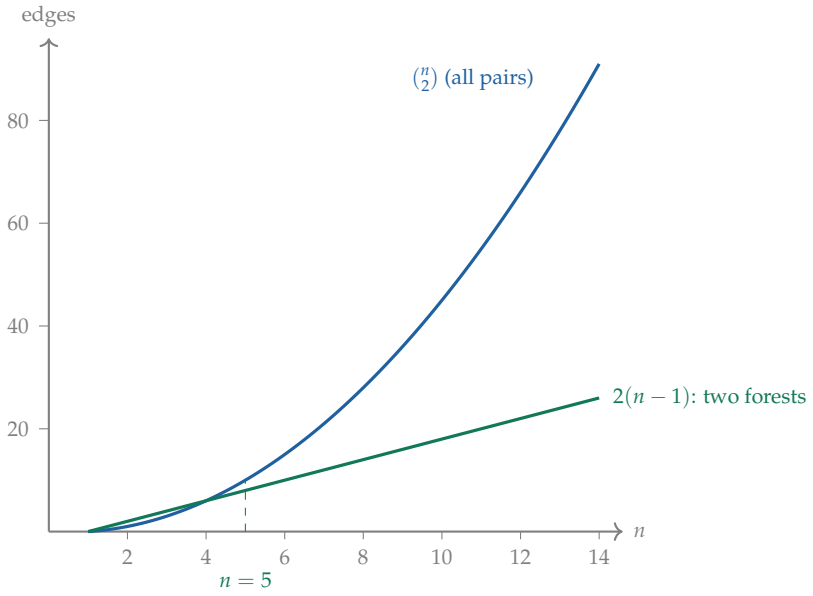


Figure 1. The edge budget. Once the blue curve (all pairs of K_n) is above the line "twice the cap", G and $\overline{G}$ cannot both have the property. Green: forests, from $n = 5$ on.

Simple example: a graph and its complement cannot both be forests when $n \geq 5$

A forest (a graph with no cycle) on n vertices has at most $n-1$ edges. If G and $\overline{G}$ were both forests, then

$$\binom{n}{2} = e(G) + e(\overline{G}) \leq 2(n-1) \iff n(n-1) \leq 4(n-1) \iff n \leq 4.$$

So for $n \geq 5$ at least one of G, $\overline{G}$ contains a cycle. Check with $G = K_{1,4}$: it is a tree, and its complement $K_4 \cup K_1$ is full of triangles (Figure 2). ✓

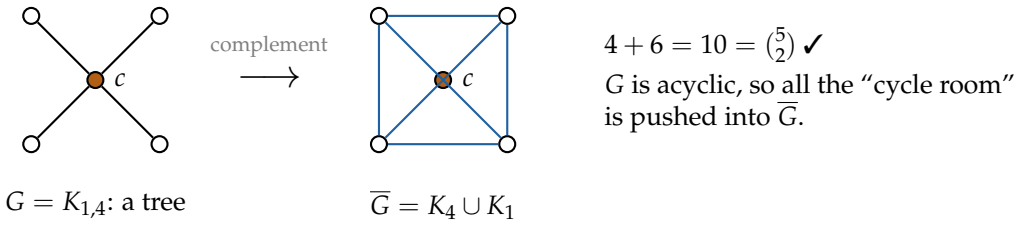


Figure 2. On 5 vertices a forest and its complement cannot both be acyclic.

The edge-budget lemma, and the caps you know

Lemma. Let $\mathcal{P}$ be a property such that every graph on n vertices with $\mathcal{P}$ has at most $f(n)$ edges. If $\binom{n}{2} > 2f(n)$, then for every graph G on n vertices, G or $\overline{G}$ fails $\mathcal{P}$.
Proof. $e(G) + e(\overline{G}) = \binom{n}{2}$; if both had $\mathcal{P}$ the left side would be at most $2f(n)$. ■
Caps. Forest: $e \leq n-1$. Simple planar, $n \geq 3$: $e \leq 3n-6$. Simple planar and triangle-free (for example bipartite), $n \geq 3$: $e \leq 2n-4$.

Comprehensive example: when counting is not enough

Claim. For $n \geq 5$, a graph G and its complement cannot both be bipartite.
Counting fails. A bipartite graph has at most $\lfloor n^2/4 \rfloor$ edges (Tutorial 8, A4), and $2 \cdot \frac{n^2}{4} = \frac{n^2}{2} > \binom{n}{2}$ for every n. The budget is never exceeded, so the lemma says nothing. That does *not* mean the claim is false; it means we need structure.
Structure. Suppose G is bipartite with parts X, Y. Since $|X| + |Y| = n \geq 5$, one part, say X, has at least 3 vertices. No two vertices of X are adjacent in G, so any three of them are pairwise adjacent in $\overline{G}$: a triangle. A triangle is an odd cycle, so $\overline{G}$ is not bipartite (Section 1.5). ■

A trap

The lemma only works in one direction: a budget that is *not* exceeded proves nothing (the bipartite example above).

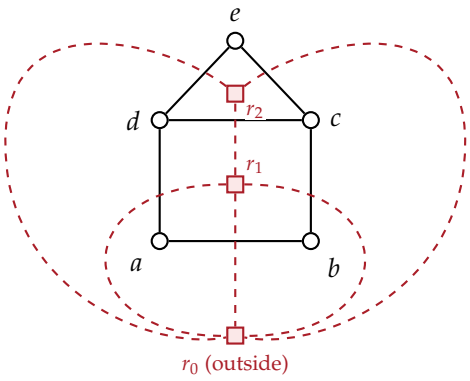
Edge budget in one breath

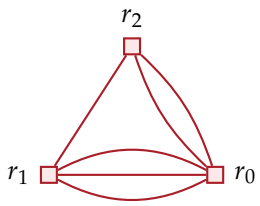
$e(G) + e(\overline{G}) = \binom{n}{2}$. If a property caps edges at $f(n)$ and $\binom{n}{2} > 2f(n)$, then G or $\overline{G}$ lacks the property. When the budget is not exceeded, look for a structural argument instead.

1.2 Dual graphs

The intuition

Take a planar *drawing* of G. Put a new red vertex inside every region (one in the infinite region too). For every edge of G, draw a red edge that crosses it once and joins the red vertices of the two regions on either side. The red graph is the **dual** G^d. It swaps roles: **regions of** G **become vertices of** G^d, every edge stays an edge (just rotated by a right angle), and, for a connected G, the vertices of G become the regions of G^d. The degree of a region in G is the degree of its red vertex in G^d.





G^d redrawn: 3 vertices, 6 edges, a *multigraph*

Figure 3. The "house" G (black, $v = 5$, $e = 6$, $r = 3$) and its dual (red). Region degrees $\deg r_1 = 4$, $\deg r_2 = 3$, $\deg r_0 = 5$ become vertex degrees in G^d, and $4 + 3 + 5 = 12 = 2e$. Three edges (ab, bc, da) separate r_1 from r_0, so G^d has three parallel edges there.

Simple example: the dual of a triangle

K_3 drawn in the plane has 2 regions, inside and outside, and each of its 3 edges separates them. So K_3^d has 2 vertices joined by 3 parallel edges. Even the dual of a simple graph can be a **multigraph**.

Dual of a plane graph

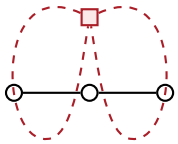
Let G be a connected graph with a fixed planar embedding having regions $R_1, \ldots, R_r$. The **dual** G^d has one vertex x_i for each region R_i, and for each edge ϵ of G one edge ϵ^d joining the vertices of the two regions on the two sides of ϵ (a **loop** if both sides are the same region, which happens exactly when ϵ is a bridge). Then

$$|V(G^d)| = r, \qquad |E(G^d)| = e, \qquad \deg_{G^d}(x_i) = \deg(R_i), \qquad G^d \text{ is planar with } v \text{ regions.}$$

(The last one is Euler's formula applied to G^d, which is connected: $r - e + r^d = 2$ and $v - e + r = 2$ give $r^d = v$.) The dual depends on the embedding, not only on the graph.

Comprehensive example: the cube and the octahedron

The cube Q_3 is connected and planar with $v = 8$, $e = 12$, $r = 6$, and every region is a quadrilateral. Its dual has a vertex for each of the 6 faces; two faces share an edge exactly when they are not opposite, so each dual vertex has degree 4 (matching $\deg R = 4$). That is the octahedron $K_{2,2,2}$: 6



a path: 1 region; the dual is one vertex with 2 loops

	v	e	r
cube Q_3	8	12	6
octahedron	6	12	8

dual of the cube = octahedron: v and r swap, e stays

Figure 4. Left: a bridge has the same region on both sides, so it becomes a loop. Right: the cube and the octahedron are duals of each other.

vertices, 12 edges, and by Euler $r^d = 12 - 6 + 2 = 8$, the number of vertices of the cube. ✓ In general, *"every region of G has degree k"* and *"G^d is k-regular"* say the same thing.

Common confusions

(i) The dual has as many **edges** as G, not as many vertices.
(ii) The dual of a simple graph need not be simple (triangle: parallel edges; bridge: loop).
(iii) The infinite region gets a dual vertex too; forgetting it loses all the edges on the outer boundary.

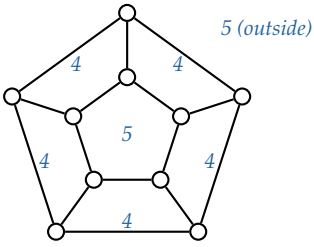
Duals in one breath

Regions ↔ vertices, edges ↔ edges, (connected) vertices ↔ regions; region degree = dual vertex degree; bridges give loops.

1.3 Counting with region degrees

The intuition

Tutorial 8 used "every region has at least 3 sides" to get $e \leq 3v - 6$. The same machine accepts any information about region sizes. If every region has degree at least k, then counting edge-sides gives $kr \leq 2e$: there are only $2e$ edge-sides to share out. Put that into $v - e + r = 2$ and you can bound whichever of v, e, r the question asks about. The only extra care is at the end: the bound you get is usually a fraction, and the quantity is an integer, so you **round in the right direction**.



pentagonal prism: $v = 10$, $e = 15$, $r = 7$

Region degrees: $5 + 5 \cdot 4 + 5 = 30 = 2e$. ✓

Every region has degree ≥ 4, so $4r \leq 2e$: $28 \leq 30$. ✓

Careful: "$\deg(R)$" counts edge-*sides*. An edge that sticks into a region (a bridge) is counted twice, so $\deg(R)$ $\geq$ (number of distinct boundary edges).

Figure 5. Region degrees of the pentagonal prism. Each edge has two sides, and each side belongs to exactly one region.

Simple example: long boundaries mean few regions

A connected planar graph has $e = 31$ edges and every region has degree at least 6. Then $6r \leq 2e = 62$, so $r \leq 10.\overline{3}$ and, r being an integer, $r \leq 10$. Euler then gives $v = 2 + e - r \geq 2 + 31 - 10 = 23$.

Region-degree counting

Let G be a connected planar graph with an embedding with r regions. If $\deg(R) \geq k$ for every region R, then

$$k\,r \;\leq\; \sum_{R} \deg(R) \;=\; 2e,$$

and combining with $v - e + r = 2$ (eliminate r):

$$e \;\leq\; \frac{k}{k-2}\,(v-2) \qquad (k \geq 3).$$

Rounding. If you derive $x \geq 30.5$ for an integer x, conclude $x \geq 31$ (round *up* a lower bound, *down* an upper bound). Always round the quantity you have just bounded, before feeding it on.

Comprehensive example: why a football needs pentagons

Claim. There is no connected planar graph in which every vertex has degree ≥ 3 *and* every region has degree ≥ 6.

Proof. Vertex degrees give $3v \leq \sum_x \deg(x) = 2e$; region degrees give $6r \leq 2e$. Then

$$2 = v - e + r \;\leq\; \frac{2e}{3} - e + \frac{2e}{6} = 0,$$

a contradiction. ■

So a ball stitched from hexagons with three seams at each corner is impossible; the classic football uses 12 pentagons to make the count work. (Repeat the computation allowing p pentagonal regions and all other regions of degree ≥ 6: you get $p \geq 12$.)

Direction of the inequalities

"Every region has at least k edges" gives $kr \leq 2e$, **not** $2e \leq kr$. Sanity check with the cube: $k = 4$, $4 \cdot 6 = 24 \leq 24 = 2e$. And write the integer rounding as its own line.

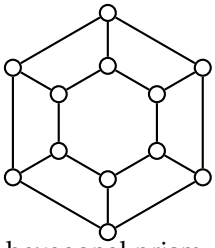
Region counting in one breath

$\sum_R \deg(R) = 2e$; "each region $\geq k$" $\Rightarrow kr \leq 2e$; feed into $v - e + r = 2$; round the integer the right way.

1.4 Degree-sum extremal problems: bound, then attain

The intuition

"What is the largest (or smallest) possible ..." has **two** halves, and both are compulsory. The Handshaking Theorem usually gives one half in a line: $\sum \deg = 2e$, so a minimum degree δ forces $\delta v \leq 2e$. The other half is a **construction**: one explicit graph that meets the bound and satisfies every condition in the question (connected, simple, the right number of edges, ...). Without it, you have only proved "at most", and the true answer could be smaller.



hexagonal prism:
12 vertices, all of degree 3, 18 edges

Bound. 12 vertices of degree ≥ 3: $2e \geq 36$, so $e \geq 18$.

Attain. The hexagonal prism is connected, simple, every degree is 3, and $e = 18$.

So the minimum is exactly 18: both halves are done.

Figure 6. A bound from the Handshaking Theorem, and a graph that shows it cannot be improved.

Simple example: fewest edges with 12 vertices of degree at least 3

Bound: $2e = \sum \deg \geq 3 \cdot 12 = 36$, so $e \geq 18$. Attained: the hexagonal prism (Figure 6). Answer: 18.

The two halves

For a graph with v vertices, e edges and minimum degree δ:

$$\delta v \leq 2e, \qquad \text{so} \qquad v \leq \left\lfloor \frac{2e}{\delta} \right\rfloor \quad \text{and} \quad e \geq \left\lceil \frac{\delta v}{2} \right\rceil.$$

To show a bound is the answer, exhibit a graph that attains it and check *every* hypothesis. Useful parts: cycles (2-regular), prisms (3-regular), and "add a few chords to a cycle" to adjust degrees one at a time.

Comprehensive example: 7 vertices, every degree at least 3

Bound. $2e \geq 3 \cdot 7 = 21$, and $2e$ is even, so $2e \geq 22$ and $e \geq 11$. (A 3-regular graph on 7 vertices is impossible: $7 \cdot 3$ is odd.)
Attain. We need degrees summing to 22 with all ≥ 3: six vertices of degree 3 and one of degree 4. Take the 7-cycle $0, 1, \ldots, 6$ and add the chords $\{0,2\}$, $\{0,5\}$, $\{1,4\}$, $\{3,6\}$. Degrees: vertex 0 has 4, every other vertex has 3; $7 + 4 = 11$ edges; connected because it contains the cycle. Answer: 11.

Do not stop at the bound

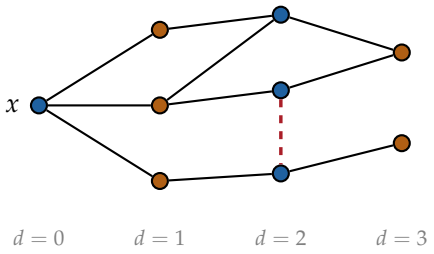
"$v \leq 2e/\delta$, so the maximum is $\lfloor 2e/\delta \rfloor$" is only half a proof. Parity, simplicity or connectedness can make a bound unreachable (with 5 vertices of degree ≥ 4 and 11 edges the bound $\lfloor 22/4 \rfloor = 5$ is fine, but with 5 vertices and $e = 11 > \binom{5}{2} = 10$ no simple graph exists at all), and integrality

forces rounding (in the 7-vertex example, $\lceil 21/2 \rceil = 11$ is justified by evenness). Build the graph.

1.5 Bipartite graphs and odd cycles

The intuition

A graph is bipartite when you can colour its vertices with two colours so that every edge joins different colours. To *find* such a colouring, pick a vertex, colour it blue, colour its neighbours orange, their new neighbours blue, and so on, layer by layer (breadth-first search). The only way this can fail is an edge inside one layer, which closes an odd cycle.



Colour by the parity of the distance d from x.
Edges between consecutive layers always cross colours.
The dashed red edge sits *inside* layer 2: with the two shortest paths back to x it closes a closed walk of odd length $2 + 2 + 1 = 5$, which contains an odd cycle.

Figure 7. Two-colouring by distance layers. Without the dashed edge the graph is bipartite; with it, it is not.

Simple example: colouring an even cycle and a cube

C_6 with vertices $0, \ldots, 5$: colour i blue if i is even, orange if odd; each edge $\{i, i+1\}$ and the closing edge $\{5, 0\}$ join different colours. Q_3: colour a bit string by the parity of its number of 1s; every edge flips one bit, so it joins different colours (Tutorial 8).

Bipartite

$G = (V, E)$ is **bipartite** if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge has one end in V_1 and one in V_2. Equivalently, G has a proper 2-colouring.
Theorem (Kőnig). If G contains no cycle of odd length, then G is bipartite.
Proof (it suffices to colour each component). Fix a vertex x in a component and put $V_1 = \{u : d(x,u) \text{ even}\}$, $V_2 = \{u : d(x,u) \text{ odd}\}$. If an edge uw had $d(x,u)$ and $d(x,w)$ of the same parity, then a shortest x–u path, the edge uw, and a shortest w–x path form a closed walk of odd length $d(x,u) + 1 + d(x,w)$. An odd closed walk always contains an odd cycle (take a shortest one; if it repeated a vertex it would split into two shorter closed walks, one of them odd). Contradiction. ■

Comprehensive example: trees and grids are bipartite

Trees. A tree has no cycles at all, so in particular no odd cycle: by Kőnig's theorem it is bipartite. Concretely, colour by the parity of the distance from a fixed root.
Grids. In the $m \times n$ grid, vertices (i, j) and edges between horizontal or vertical neighbours, every edge changes $i + j$ by exactly 1. So "$i + j$ even" versus "$i + j$ odd" is a bipartition; the colouring is the chessboard.

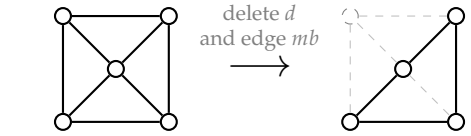
Produce the parts

To prove "bipartite", name V_1 and V_2 and check every edge crosses. "It looks bipartite" or "I could colour it" is not a proof.

1.6 Properties inherited by subgraphs: restrict the witness

The intuition

Many properties are **certified by a witness**: a crossing-free drawing certifies planarity, a list of colours certifies colourability, a bound on every degree certifies "maximum degree $\leq d$". If you delete vertices or edges, you can simply **keep the old witness** and throw away the parts that no longer exist: erase the deleted edges from the drawing, forget the colours of deleted vertices. Nothing can go wrong, because you removed things and added nothing. Properties of this kind are called **hereditary** (closed under taking subgraphs).



G (a wheel), drawn without crossings

the same drawing, minus what was deleted

Hereditary: planar, forest, triangle-free, maximum degree $\leq d$, k-colourable.

Not hereditary: connected, Hamiltonian, "has a cycle", "minimum degree ≥ 2". Deleting edges can destroy these.

Figure 8. A subgraph of a plane drawing is drawn by erasing. The witness is restricted, never rebuilt.

Simple example: subgraphs of planar graphs are planar

Let H be a subgraph of a planar graph G. Take a crossing-free drawing of G and erase every vertex and edge not in H. What remains is a drawing of H, and it has no crossings because the original had none. So H is planar.

Subgraphs, and the restriction pattern

$H = (V_H, E_H)$ is a **subgraph** of $G = (V, E)$ if $V_H \subseteq V$, $E_H \subseteq E$, and every edge in E_H has both ends in V_H. A property $\mathcal{P}$ is **hereditary** if every subgraph of a graph with $\mathcal{P}$ has $\mathcal{P}$.
Proof pattern. (1) Take the witness for G. (2) Restrict it to H (intersect every set with V_H, keep only the edges in E_H). (3) Check the defining condition for H; it holds because every object of H is an object of G.

Comprehensive example: triangle-free, bounded degree

Triangle-free. A triangle in H uses three edges of $E_H \subseteq E$, so it would be a triangle in G.
Maximum degree $\leq d$. For $x \in V_H$, every edge of H at x is an edge of G at x, so $\deg_H(x) \leq \deg_G(x) \leq d$. Note what made it work: we never had to invent anything new for H.

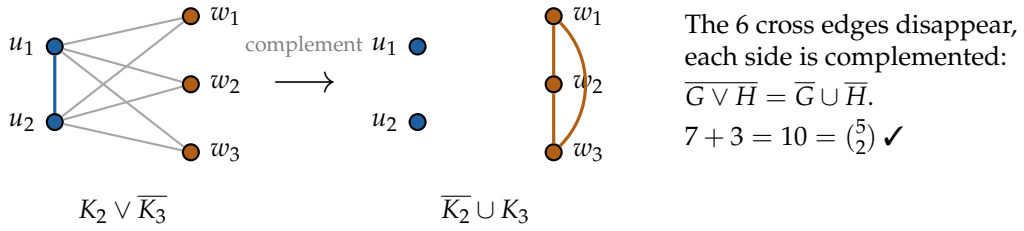
Hereditary is special

"Connected" is the classic non-example: delete the middle vertex of a path and the subgraph falls apart. Before using "subgraphs inherit $\mathcal{P}$", make sure $\mathcal{P}$ is of the witness-restriction kind.

1.7 Joins, complements and connectivity

The intuition

The **disjoint union** $G \cup H$ puts two graphs side by side with nothing between them. The **join** $G \vee H$ puts them side by side and adds *every* edge between them. Complementing swaps "all" and "nothing", so the complement of a join is a disjoint union of the complements. This little identity is the main tool for *building* graphs whose complements you control.



The 6 cross edges disappear, each side is complemented:
$\overline{G \vee H} = \overline{G} \cup \overline{H}$.
$7 + 3 = 10 = \binom{5}{2}$ ✓

Figure 9. The complement of a join is the disjoint union of the complements. The join is connected; its complement has 3 *components.*

Simple example: wheels and complete bipartite graphs

The wheel W_n (a hub joined to every vertex of a cycle C_n) is $C_n \vee K_1$, so $\overline{W_n} = \overline{C_n} \cup K_1$: the hub is isolated in the complement. And $K_{m,n} = \overline{K_m} \vee \overline{K_n}$, so $\overline{K_{m,n}} = K_m \cup K_n$, as found in Tutorial 8.

Union, join, and their complements

For graphs G, H on disjoint vertex sets:

$$G \cup H = (V_G \cup V_H,\ E_G \cup E_H), \qquad G \vee H = \big(V_G \cup V_H,\ E_G \cup E_H \cup \{\{g,h\} : g \in V_G,\ h \in V_H\}\big).$$

Then $\overline{G \vee H} = \overline{G} \cup \overline{H}$. A join of two non-empty graphs is **connected**, with diameter at most 2: two vertices on different sides are adjacent, and two on the same side have a common neighbour on the other side.

Comprehensive example: reading off a complement

Let G be the path $a - b - c$ joined to two new vertices x, y that are adjacent to each other, i.e. $G = P_3 \vee K_2$ (5 vertices, $2 + 1 + 6 = 9$ edges). Then

$$\overline{G} = \overline{P_3} \cup \overline{K_2},$$

where $\overline{P_3}$ is the single edge ac plus the isolated vertex b, and $\overline{K_2}$ is two isolated vertices. So $\overline{G}$ has exactly one edge, ac, and $9 + 1 = 10 = \binom{5}{2}$. ✓ Without the identity you would have to check all ten pairs one by one.

Complements and connectivity

The complement of a connected graph can be disconnected (Figure 9, or $K_{1,3}$ whose complement is $K_3 \cup K_1$). Connectedness of G tells you nothing directly about $\overline{G}$; the join identity is how you see what happens.

Joins in one breath

$\overline{G \vee H} = \overline{G} \cup \overline{H}$; joins are connected with diameter ≤ 2; complementing turns "all edges between the parts" into "no edges between the parts".

The toolbox at a glance

Situation	Tool
G and $\overline{G}$ both have property $\mathcal{P}$?	edge budget: if $\binom{n}{2} > 2f(n)$, one of G, $\overline{G}$ fails $\mathcal{P}$
Regions $\leftrightarrow$ vertices	dual: $v^d = r$, $e^d = e$, $r^d = v$, $\deg_{G^d} = \deg(R)$
Information about region sizes	$kr \leq 2e$, then Euler; round integers correctly
"Largest / smallest possible"	Handshaking bound *and* an explicit graph that attains it
Prove bipartite	name V_1, V_2; or show no odd cycle (Kőnig)
Do subgraphs keep the property?	restrict the witness (drawing, edge list, degree bound)
Complement of a structured graph	$\overline{G \vee H} = \overline{G} \cup \overline{H}$

2 Mistakes and "spot the error"

Mistakes that cost marks

1. Concluding something from a budget or bound that is *satisfied*. A satisfied bound proves nothing.
2. Reversing $kr \leq 2e$, or rounding an integer bound the wrong way ($x \geq 41.5 \Rightarrow x \geq 42$, not 41).
3. Stopping at the bound in a "maximum" or "minimum" question, with no graph that attains it.
4. Forgetting the infinite region when building a dual, or claiming the dual of a simple graph is simple.
5. Asserting "this property passes to subgraphs" for a property that does not (connected, Hamiltonian).
6. Proving existence by showing that a necessary condition holds ("the edge count is an integer, so the graph exists"). Existence needs a construction.
7. Saying two graphs are isomorphic "easily" without writing the bijection.

Spot the error 1

"*Each region has at least* 4 *edges, so* $2e \leq 4r$*; with* $e = 20$ *this gives* $r \geq 10$."

Spot the error 2

"We found $v \geq 41.5$. Since v is an integer, $v \geq 41$."

Spot the error 3

"G has 7 vertices and 10 edges, so $e \leq 3v - 6 = 15$ holds, and $\overline{G}$ has $11 \leq 15$ edges, so both G and $\overline{G}$ are planar."

Spot the error 4

"Every subgraph of a connected graph is connected, because we only delete things."

Spot the error 5

"K_4 is a simple planar graph, so its dual is a simple graph with 4 vertices." (Part of this is true. Which part, and why is the reasoning wrong?)

3 Practice problems

Try each one for ten minutes before you look at the appendix.

E1. Prove that for $n \geq 6$, a graph on n vertices or its complement contains a triangle. (Hint: look at one vertex and its ≥ 5 neighbours or non-neighbours.)

E2. Draw the dual of the triangular prism (two triangles joined by three edges). How many vertices, edges and regions does it have, and which familiar solid is it?

E3. Prove that if a connected plane graph has a bridge, its dual has a loop.

E4. A connected planar graph is 4-regular and every region is a triangle. Find v, e and r, and name the graph.

E5. A connected planar graph has 10 vertices, all of degree 3. How many regions does every embedding have? Can every region be a pentagon?

E6. A graph has 10 vertices and every vertex has degree at least 5. What is the smallest possible number of edges? Prove both halves.

E7. Prove that every subgraph of a forest is a forest, and give an example showing that a subgraph of a tree need not be a tree.

E8. Show that the complement of the wheel W_6 (hub plus 6-cycle) is planar.

E9. A plane graph has 9 edges and 5 regions and is connected. How many vertices, edges and regions does its dual have?

A Answers to the "spot the error" puzzles

1. The inequality is reversed. "At least 4 edges per region" gives $4r \leq 2e$, so $r \leq 10$: an *upper* bound on r.

2. Wrong rounding direction. $v \geq 41.5$ and v an integer give $v \geq 42$; 41 violates $v \geq 41.5$.

3. A satisfied bound proves nothing about planarity ($K_{3,3}$ satisfies $e \leq 3v - 6$). Both graphs may well be planar, but you must draw them, or find a Kuratowski subgraph to show one is not.

4. False: deleting the middle vertex of the path $a - b - c$ leaves two isolated vertices. "Connected" is not hereditary; deleting can destroy connections.

5. The conclusion is true, the reason is not. K_4 has $r = 4$ regions, all triangles, and any two of them share exactly one edge, so the dual is K_4 again, which is simple. But "simple, so the dual is simple" is false in general: the dual of K_3 has three parallel edges, and a bridge gives a loop.

B Answers to the practice problems

E1. Take a vertex x. Among the other $n - 1 \geq 5$ vertices, at least 3 are neighbours of x in G, or at least 3 are non-neighbours (pigeonhole). In the first case, if two of those 3 are adjacent in G they form a triangle with x; if not, the 3 are pairwise adjacent in $\overline{G}$, a triangle. The second case is the same argument with G and $\overline{G}$ swapped.

E2. The prism has $v = 6$, $e = 9$, $r = 5$ (two triangles, three quadrilaterals). The dual has 5 vertices and 9 edges: the two triangle-vertices have degree 3 and are each joined to all three quadrilateral-vertices, which form a triangle among themselves. By Euler it has $9 - 5 + 2 = 6$ regions. It is the **triangular bipyramid** (two tetrahedra glued along a face).

E3. *Sketch.* Let $\epsilon = xy$ be a bridge and let A be the component of $G - \epsilon$ that contains y. Because ϵ lies on no cycle, nothing fences off one side of ϵ from the other: start just to the left of ϵ, walk round the drawing of A keeping close to it, and you arrive just to the right of ϵ without crossing any edge. So both sides of ϵ lie in the same region, and ϵ^d joins that region's vertex to itself: a loop. (Conversely, an edge on a cycle has the inside of the cycle on one side and the outside on the other, so its two sides are different regions.)

E4. $2e = 4v$ and $2e = 3r$, so $e = 2v$ and $r = \frac{4v}{3}$. Euler: $v - 2v + \frac{4v}{3} = 2$, so $v = 6$, $e = 12$, $r = 8$: the **octahedron**.

E5. $e = \frac{10 \cdot 3}{2} = 15$, so $r = 15 - 10 + 2 = 7$. If every region were a pentagon then $\sum \deg R = 5 \cdot 7 = 35 \neq 30 = 2e$. No. (The pentagonal prism of Figure 5 has 2 pentagons and 5 quadrilaterals: $10 + 20 = 30$.)

E6. Bound: $2e = \sum \deg \geq 5 \cdot 10 = 50$, so $e \geq 25$. Attained: vertices $0, \ldots, 9$ with $i \sim j$ exactly when $j - i \equiv \pm 1, \pm 2$ or $5 \pmod{10}$. Each vertex has $2 + 2 + 1 = 5$ neighbours, so the graph is 5-regular with $\frac{10 \cdot 5}{2} = 25$ edges. So the minimum is 25.

E7. A cycle in a subgraph would be a cycle in the forest, so subgraphs of forests are forests (restrict the witness "no cycle"). Deleting an edge of a tree gives a forest with two components, which is not a tree.

E8. $\overline{W_6} = \overline{C_6} \cup K_1$ (Section 1.7), and $\overline{C_6}$ is the triangular prism (Tutorial 9), which is planar. An isolated vertex can be drawn anywhere.

E9. By Euler $v = 2 + e - r = 2 + 9 - 5 = 6$. The dual has $r = 5$ vertices, $e = 9$ edges, and $v = 6$ regions.