Tutorial 3: Shelves, Round Tables, Card Hands, Multinomials and Slack Variables

Books on shelves, round tables, card hands, the multinomial theorem, and inequalities with slack variables

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count arrangements of distinct objects on shelves by **lining them up and then cutting the row**, with and without the condition "at least one on each shelf".
2. Count seatings at a **round table** with $(n-1)!$, and combine this with the gap method (a circle of m people has m gaps, not $m+1$).
3. Count card hands by choosing the **ranks first and then the suits**, and decide correctly between ordered ranks (different roles) and a set of ranks (the same role).
4. Count strings with a prescribed number of each symbol with the **multinomial coefficient** $\binom{n}{n_1,\ldots,n_k}$, and split "at least" and "weight" conditions into exact cases.
5. Read off coefficients with the **multinomial theorem**, including numerical coefficients, signs and the constant term, and evaluate sums by **plugging values into the binomial theorem**.
6. Count integer solutions of an **inequality** with a slack variable, and handle **negative lower bounds** by shifting the variables.

Sections 1.1 to 1.7 review the tools with the intuition, a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 collects the classic mistakes, "spot the error" puzzles and a checklist, and Section 3 has practice exercises; the puzzles are answered in the appendix, so the notes work for **self-study**. Tutorials 1 and 2 (sum and product rules, $P(n,r)$, $\binom{n}{r}$, the gap method, stars and bars) are assumed.

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1 The toolbox

Every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap.

1.1 Books on shelves: line them up, then cut

The intuition

Placing distinct books on shelves, where the order on each shelf matters, looks like a two-step choice (which books go where, and in what order). There is a much faster way to see it: **read the shelves one after the other**. Shelf 1 from left to right, then shelf 2 from left to right, gives **one long row of books**, together with the place where shelf 1 ends. So a placement is the same thing as a *row of all the books* plus a *cut* in the row.

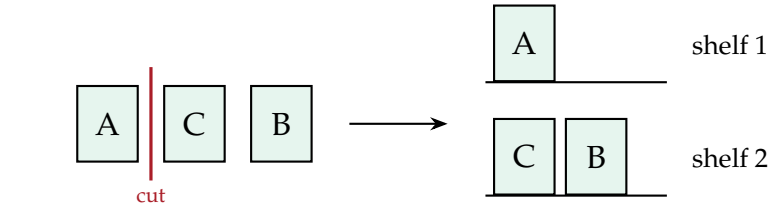


Figure 1. The row ACB with a cut after the first book is the placement "A on shelf 1; C then B on shelf 2". Reading the shelves in order gives the row back, so no placement is counted twice.

Simple example: four books, three shelves

Place the books A, B, C, D on three shelves with at least one book on each shelf. Line the books up: $4! = 24$ rows. Each row has 3 gaps between consecutive books, and we need 2 cuts in two different gaps so that no shelf is empty: $\binom{3}{2} = 3$ ways. By the rule of product there are $24 \cdot 3 = 72$ placements.
Check by listing. One shelf gets two books and the others one each: choose that shelf (3), its two books ($\binom{4}{2} = 6$) in some order (2), and the order in which the other two books go to the other two shelves (2): $3 \cdot 6 \cdot 2 \cdot 2 = 72$. ✓

Distinct objects on labelled shelves

Let $n \geq 1$ distinct objects be placed on k labelled shelves, where the left-to-right order on each shelf matters.

(i) **Every shelf non-empty.** Arrange the objects in a row, then choose $k-1$ of the $n-1$ gaps *between* consecutive objects for the cuts (never an end, never two cuts in one gap), and use the rule of product.

(ii) **Empty shelves allowed.** Now two cuts may be in the same place, or at an end of the row. Treat the $k-1$ cuts as identical dividers and arrange the n distinct objects together with them:

$$\frac{(n+k-1)!}{(k-1)!}.$$

Both counts work because reading shelf 1, then shelf 2, ..., then shelf k recovers the row and the

positions of the cuts: each placement corresponds to exactly one row with cuts.

Comprehensive example: six books on three shelves

Six different books are placed on three labelled shelves. (a) How many placements if shelves may be empty? (b) How many if every shelf gets at least one book?

Solution. (a) Arrange 6 books and 2 identical dividers: $\frac{8!}{2!} = 20\,160$. (b) Arrange the books (6!) and choose 2 of the 5 inner gaps for the cuts: $6!\binom{5}{2} = 720 \cdot 10 = 7200$.
Check (b) with inclusion-exclusion. From the 20 160 placements of (a), remove those with an empty shelf. If a given shelf is empty, the books sit on the other two shelves in $\frac{7!}{1!} = 5040$ ways; there are 3 choices of the empty shelf. If two given shelves are empty, all books are on the third shelf: $6! = 720$ ways, 3 choices. So $20\,160 - 3 \cdot 5040 + 3 \cdot 720 = 7200$. ✓

Do not forget the order on the shelves

"Each book chooses a shelf" gives k^n and only says *which* books are on which shelf. The books on a shelf can still be arranged among themselves, and those arrangements are different placements. Choosing the subsets is only half of the count.

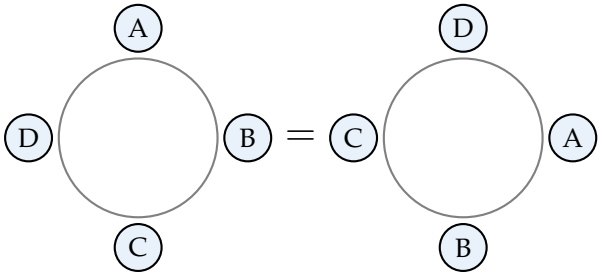
Shelves in one breath

- Placement on shelves = a row of all the objects + cuts. Non-empty shelves: cuts in distinct inner gaps. Empty allowed: $\frac{(n+k-1)!}{(k-1)!}$.

1.2 Round tables: fix one person

The intuition

At a round table, what matters is **who sits next to whom**, not which chair someone took: if everybody moves one seat to the left, nothing has changed. The easy way to remove this rotation is to **seat one person anywhere and use them as the reference point**. Everybody else is then placed relative to that person, exactly as in a row.



Same neighbours for everyone:
4 rotations give 1 seating.
4! rows $\rightarrow$ $\frac{4!}{4} = 3! = 6$ seatings.

Figure 2. Rotating everybody gives the same circular seating. Each seating corresponds to n rows (cut the circle just before any of the n people), so there are $n!/n = (n-1)!$ seatings.

Simple example: four friends at a round table

Seat A anywhere. Going clockwise from A, the other three people can be arranged in $3! = 6$ ways, and different clockwise orders really are different seatings (someone has a different neighbour). So there are 6 seatings, compared with $4! = 24$ in a row. Each seating has been counted 4 times in the row count, once for each person who could sit "first".

Circular arrangements and the circular gap method

(i) The number of ways to seat n distinct people around a round table, where two seatings are the same if one is a rotation of the other, is $\frac{n!}{n} = (n-1)!$.

(ii) If the seats are **numbered** (rotations count as different), the number is $n!$, as in a row.

(iii) **Gap method on a circle.** $m \geq 2$ people seated around a table create exactly m gaps (one between each pair of neighbours), because the gap "after the last person" and the gap "before the first person" are the same gap. In a row, m people create $m+1$ gaps.

Comprehensive example: study group around a table

Five CS students and three Math students (all different) sit at a round table with 8 seats. In how many seatings are no two Math students next to each other?

Solution. First seat the 5 CS students around the table: $(5-1)! = 4! = 24$ ways. They create 5 gaps. Two Math students are neighbours exactly when they are in the same gap, so we choose 3 of the 5 gaps, $\binom{5}{3} = 10$ ways, and place the 3 Math students in them, $3! = 6$ ways. By the rule of product: $24 \cdot 10 \cdot 6 = 1440$.

Check. Choosing 3 gaps and then ordering the Math students is the same as placing them in 3 distinct gaps one by one: $P(5,3) = 60$, and $24 \cdot 60 = 1440$. ✓

Two classic slips at a round table

- Using $n!$ for the first group: this counts rotations as different, but the question treats them as the same. Once one group is seated around the table, the rotation is fixed, so the second group is placed with ordinary (non-circular) counting.
- Using the *row* number of gaps, $m+1$. On a circle the two end gaps are the same gap, and two people placed "at both ends" would sit next to each other.

Round tables in one breath

- n people around a table: $(n-1)!$. Numbered seats: $n!$.
- Keep a group apart: seat the others around the table first ($(m-1)!$), then choose among the m gaps.

1.3 Card hands: ranks first, then suits

The intuition

A standard deck has 52 cards: 13 **ranks** (A, 2, 3, …, 10, J, Q, K) in each of 4 **suits** (♠, ♡, ♢, ♣). A card is a pair (rank, suit). Most hands are described by the *pattern of ranks*: "three of one rank and three of another" is a pattern for a six-card hand. So count in two stages: **choose the ranks**

that play each role in the pattern, **then choose the suits** (which cards of each chosen rank).

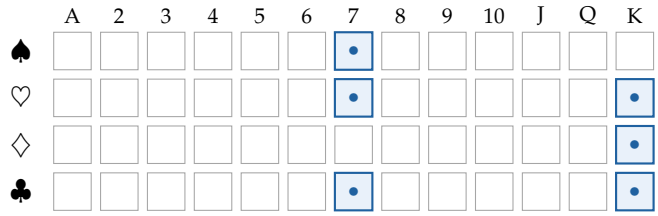


Figure 3. The deck as a 4×13 *grid (suits* $\times$ *ranks). The highlighted six-card hand is "two triples": three 7's and three kings. First the ranks are chosen (columns), then the suits inside each chosen column. Both columns are coloured the same because the two ranks play the same role.*

Simple example: two-card hands

A two-card hand is either a *pair* (both cards of one rank) or two cards of *different ranks*.

- Pair: choose the rank (13), then 2 of its 4 suits ($\binom{4}{2} = 6$): $13 \cdot 6 = 78$.
- Different ranks: the two ranks play the *same* role, so choose them together: $\binom{13}{2} = 78$; then one suit for each: $4 \cdot 4$. Total $78 \cdot 16 = 1248$.

Check: $78 + 1248 = 1326 = \binom{52}{2}$, the number of all two-card hands. ✓

Counting a hand by its rank pattern

Suppose a hand type is described by a pattern of ranks, such as $3+3$ or $4+2$ (six-card hands) or $2+1+1+1$ (a five-card hand with one pair), where different numbers mean different ranks. Then

$$\#\text{hands} \;=\; \Big(\#\text{ways to choose the ranks for the roles}\Big) \times \prod_{\text{chosen ranks}} \binom{4}{\#\text{cards of that rank}}.$$

- Ranks that play **different roles** (in $4+2$: the rank of the four and the rank of the pair) are chosen **in order**: $13 \cdot 12$.
- Ranks that play **the same role** (in $3+3$: the ranks of the two triples; in $2+1+1+1$: the ranks of the single cards) are chosen **as a set**: $\binom{13}{2}, \binom{12}{3}, \ldots$
- Hand names mean the **exact** pattern: "one pair" is $2+1+1+1$ with the three single cards of three further different ranks, so it contains no second pair and no triple.

Comprehensive example: one pair

How many five-card hands contain *exactly* one pair (pattern $2+1+1+1$)?

Solution. Rank of the pair: 13 ways. Ranks of the three single cards: they play the same role, and they must be different from each other and from the pair's rank, so $\binom{12}{3} = 220$ ways. Suits: $\binom{4}{2} = 6$ for the pair and 4 for each single card. By the rule of product:

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4^3 \;=\; 13 \cdot 220 \cdot 6 \cdot 64 \;=\; 1\,098\,240.$$

Ordered or unordered ranks?

The test is: *if I swap the two ranks, do I get a different hand?* In the six-card pattern $4+2$, "four 7's and two K's" is not "four K's and two 7's", so order matters ($13 \cdot 12$). With two triples ($3+3$), "7's and K's" is the same hand as "K's and 7's", so order does not matter ($\binom{13}{2}$). Getting this wrong changes the answer by a factor of exactly 2 (or 3! for three ranks).

Card hands in one breath

- Write the rank pattern, choose the ranks (ordered for different roles, a set for equal roles), then the suits.
- "Exactly" is the default: make sure the extra cards cannot upgrade the hand.

1.4 Strings with a prescribed composition

The intuition

A *ternary string* is a string of the symbols $0, 1, 2$. If we know **how many** of each symbol the string contains, counting the strings is the BANANAS problem of Tutorial 1: arrange a multiset. The new part is what to do with conditions like "at least three 1's" or "weight 3" (the *weight* of a string is the sum of its digits): **list the possible exact compositions**, count each one, and add.

Simple example: ternary strings of length 4

How many strings of length 4 have two 0's, one 1 and one 2? Choose the positions of the 0's ($\binom{4}{2} = 6$), then the position of the 1 among the remaining two ($\binom{2}{1} = 2$); the 2 goes in the last place. So $6 \cdot 2 = 12$, which is also $\dfrac{4!}{2!\,1!\,1!} = 12$.

Multinomial coefficients

Let $n = n_1 + n_2 + \cdots + n_k$ with all $n_i \geq 0$. The number of strings of length n with exactly n_i copies of symbol i (for $i = 1, \ldots, k$) is the **multinomial coefficient**

$$\binom{n}{n_1,\, n_2,\, \ldots,\, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!} = \binom{n}{n_1}\binom{n-n_1}{n_2} \cdots \binom{n-n_1-\cdots-n_{k-1}}{n_k}.$$

The product form says: choose the positions of symbol 1, then of symbol 2 among the remaining positions, and so on. For a condition that allows several compositions ("at least", "weight w"), sum the multinomial coefficient over all compositions that satisfy the condition; different compositions give different strings.

Comprehensive example: weight 3, length 6

How many ternary strings of length 6 have weight 3?

Solution. Let a be the number of 1's and b the number of 2's. The weight is $a + 2b = 3$, so $(a, b) \in \{(3, 0), (1, 1)\}$, and the remaining positions are 0's.

- $(3, 0)$: three 1's and three 0's: $\binom{6}{3} = 20$.
- $(1, 1)$: one 1, one 2 and four 0's: $\dfrac{6!}{1!\,1!\,4!} = 30$.

By the rule of sum, $20 + 30 = 50$ strings.
Check with stars and bars. A string of weight 3 is a solution of $x_1 + \cdots + x_6 = 3$ with each $x_i \in \{0, 1, 2\}$. There are $\binom{6+3-1}{3} = 56$ non-negative solutions, and exactly 6 of them use a digit 3 (not allowed). $56 - 6 = 50$. ✓

Compositions in one breath

- Exact composition $(n_1, \ldots, n_k)$: $\dfrac{n!}{n_1! \cdots n_k!}$.
- "At least", "at most", "weight": list the exact compositions, one multinomial coefficient each, then add.

1.5 The binomial and multinomial theorems

The intuition

Expanding $(a+b+c)^4 = (a+b+c)(a+b+c)(a+b+c)(a+b+c)$ means: **from each of the 4 brackets pick one term** and multiply. Every choice is a string of length 4 over $\{a, b, c\}$. The monomial a^2bc appears once for every string with two a's, one b and one c. So **the coefficient is a multinomial coefficient**, exactly the count of Section 1.4.

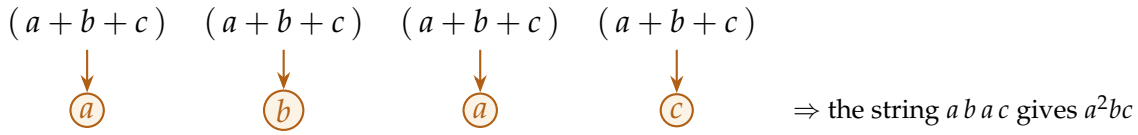


Figure 4. One term is picked from each bracket. The 12 *strings with two a's, one b and one c each produce a^2bc, so the coefficient of a^2bc in $(a+b+c)^4$ is $\frac{4!}{2!\,1!\,1!} = 12$.*

Simple example: numbers in front of the variables

The coefficient of x^2y in $(2x - 3y)^3$: the monomial comes from choosing the term $2x$ from 2 of the 3 brackets and $-3y$ from the other one. There are $\binom{3}{2} = 3$ such choices, and each contributes $(2x)^2(-3y) = -12\,x^2y$. So the coefficient is $3 \cdot 4 \cdot (-3) = -36$.

Binomial and multinomial theorems

For every integer $n \geq 0$:

(i) **Binomial theorem.** $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

(ii) **Multinomial theorem.** $(x_1 + x_2 + \cdots + x_t)^n = \displaystyle\sum_{\substack{n_1+\cdots+n_t=n \\ n_i \geq 0}} \binom{n}{n_1, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$.

(iii) **Terms with numbers.** If the brackets contain terms c_1x_1, c_2x_2, … and a constant c_0, write each term as one "block" $X_i = c_ix_i$. The coefficient of $x_1^{n_1} \cdots x_t^{n_t}$ in $(c_1x_1 + \cdots + c_tx_t + c_0)^n$ is

$$\binom{n}{n_1, \ldots, n_t, n_0} c_1^{n_1} \cdots c_t^{n_t}\, c_0^{n_0}, \qquad \text{where } n_0 = n - (n_1 + \cdots + n_t).$$

A variable that does not appear in the monomial has exponent 0 (and contributes $0! = 1$ and

$c^0 = 1$).

Comprehensive example: a constant term and a sign

Find the coefficient of x^2y^3 in $(x - 2y + 3)^7$.
Solution. The blocks are x, $-2y$ and 3. The exponents are 2 for x, 3 for y, and the constant gets the rest: $n_0 = 7 - 2 - 3 = 2$. So the coefficient is

$$\binom{7}{2,3,2} \cdot 1^2 \cdot (-2)^3 \cdot 3^2 = \frac{7!}{2!\,3!\,2!} \cdot (-8) \cdot 9 = 210 \cdot (-72) = -15\,120.$$

The three things that get lost

1. **The constant's exponent.** The exponents must add up to n, the power of the bracket, not to the degree of the monomial. The constant takes whatever is left.
2. **The numbers in the blocks.** Each number is raised to its block's exponent: $(-2y)^3$ gives $(-2)^3$, not -2.
3. **The sign.** An odd power of a negative number is negative; an even power is positive.

Multinomial theorem in one breath

- Coefficient = (number of ways to pick the blocks) $\times$ (the numbers in the blocks, to their powers).
- Exponents add up to n; the constant term is a block too.

1.6 Sums you can read off the binomial theorem

The intuition

The binomial theorem $(1 + x)^n = \sum_k \binom{n}{k} x^k$ is true **for every number** x. Plugging in a well-chosen value turns the right-hand side into a sum we want and the left-hand side into a simple number. The only work is to **recognise** $\binom{n}{k}$ **inside the sum**.

Simple example: $x = 1$ and $x = -1$

With $x = 1$: $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$. With $x = -1$: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$ for $n \geq 1$. For $n = 3$: $1 + 3 + 3 + 1 = 8 = 2^3$ and $1 - 3 + 3 - 1 = 0$. ✓

Evaluating sums with the binomial theorem

For every integer $n \geq 0$ and all numbers x, y,

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n, \qquad \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n.$$

In particular $\sum_k \binom{n}{k} = 2^n$, and $\sum_k (-1)^k \binom{n}{k} = 0$ for $n \geq 1$.

Comprehensive example: two different bases

Evaluate $S = \sum_{k=0}^{n} \binom{n}{k} 2^k 3^{n-k} (-1)^k$ for $n \geq 1$.

Solution. Group the factors that carry the exponent k: $2^k(-1)^k = (-2)^k$. Then by the binomial theorem with $x = -2$ and $y = 3$,

$$S = \sum_{k=0}^{n} \binom{n}{k} (-2)^k 3^{n-k} = (-2+3)^n = 1.$$

Check for $n = 2$: $1 \cdot 9 - 2 \cdot 2 \cdot 3 + 1 \cdot 4 = 9 - 12 + 4 = 1$. ✓

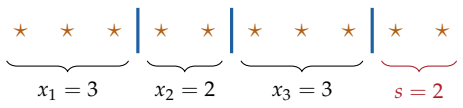
Binomial sums in one breath

- Gather everything with exponent k into x^k and everything with exponent $n-k$ into y^{n-k}, then use $(x+y)^n$.
- Always check the formula for $n = 1$ or $n = 2$.

1.7 Inequalities and lower bounds: slack variables

The intuition

Stars and bars (Tutorial 2) counts solutions of an *equation*. For an inequality like $x_1 + x_2 + x_3 \leq 10$, add one more variable, the **slack** s, that collects **what is left over**: $x_1 + x_2 + x_3 + s = 10$. Every solution of the inequality gives exactly one solution of the equation (with $s = 10 - (x_1 + x_2 + x_3) \geq 0$), and back. For a lower bound like $x_i \geq -2$, **shift the variable** so that it starts at 0.



$3 + 2 + 3 = 8 \leq 10$,
the slack takes the other 2

Figure 5. The solution $(3,2,3)$ of $x_1 + x_2 + x_3 \leq 10$ becomes the solution $(3,2,3,2)$ of $x_1 + x_2 + x_3 + s = 10$: the last part of the stars-and-bars picture belongs to the slack variable.

Simple example: $x_1 + x_2 \leq 2$

Listing the non-negative solutions: $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$, so 6 solutions. With a slack: $x_1 + x_2 + s = 2$, $s \geq 0$, has $\binom{3+2-1}{2} = \binom{4}{2} = 6$ solutions. ✓

Inequalities and shifted lower bounds

Let $n \geq 1$ and $r \geq 0$ be integers, and let all x_i be integers.

(i) **Weak inequality.** The number of solutions of $x_1 + \cdots + x_n \leq r$ with all $x_i \geq 0$ equals the number of solutions of $x_1 + \cdots + x_n + s = r$ with all variables ≥ 0: an equation in $n+1$ variables, counted by stars and bars.

(ii) **Strict inequality.** For integers, $x_1 + \cdots + x_n < r$ is the same as $x_1 + \cdots + x_n \leq r - 1$; then use (i).

(iii) **Lower bounds, also negative ones.** If $x_i \geq b_i$, put $y_i = x_i - b_i \geq 0$. Then $\sum x_i = \sum y_i + \sum b_i$, so the right-hand side changes from r to $r - \sum b_i$. A *negative* lower bound b_i *increases* the

right-hand side.

Comprehensive example: mixed lower bounds

How many integer solutions does $x_1 + x_2 + x_3 \leq 10$ have with $x_1 \geq 2$, $x_2 \geq -1$, $x_3 \geq 0$?
Solution. Put $y_1 = x_1 - 2 \geq 0$ and $y_2 = x_2 + 1 \geq 0$. Then $x_1 + x_2 + x_3 = y_1 + y_2 + x_3 + 1$, and the condition becomes $y_1 + y_2 + x_3 \leq 9$. With a slack $s \geq 0$: $y_1 + y_2 + x_3 + s = 9$, which has $\binom{4+9-1}{9} = \binom{12}{3} = 220$ solutions.
Check. Count by the exact value $t = y_1 + y_2 + x_3 \in \{0, \ldots, 9\}$: $\sum_{t=0}^{9} \binom{t+2}{2} = 1 + 3 + 6 + \cdots + 55 = 220$. ✓

Strict inequality and the direction of the shift

- For "$< r$" the slack must be at least 1 (or use "$\leq r - 1$"). A slack $s \geq 0$ with sum r also counts the solutions with sum exactly r.
- For $x_i \geq -2$ the substitution is $y_i = x_i + 2$ (so that $y_i \geq 0$). Sanity check: allowing negative values gives *more* solutions, so the answer must go *up*.

Slack variables in one breath

- $\leq r$: add $s \geq 0$. $< r$: use $\leq r - 1$, or $s \geq 1$.
- Lower bound b_i: substitute $y_i = x_i - b_i$ and replace r by $r - \sum b_i$; then stars and bars.

The toolbox at a glance

Situation	Count / method	Signal words
n distinct objects on k shelves, none empty	row, then cuts in inner gaps	"shelves", "at least one on each"
Same, shelves may be empty	$\frac{(n+k-1)!}{(k-1)!}$	"shelves", no condition
n people around a table	$(n-1)!$	"circular", "round table"
Keep a group apart around a table	$(m-1)!$, then choose among m gaps	"no two … next to each other"
Card hands	ranks (ordered or set), then suits	"pair", "triple", "same suit"
Strings with n_i copies of symbol i	$\frac{n!}{n_1! \cdots n_k!}$	"two 0's, one 1", "weight"
Coefficient in $(c_1 x_1 + \cdots + c_0)^n$	$\binom{n}{n_1, \ldots, n_0} c_1^{n_1} \cdots c_0^{n_0}$	"coefficient of", "expansion"
Sums with $\binom{n}{k} x^k y^{n-k}$	gather the powers, then $(x+y)^n$	"evaluate the sum"
$\sum x_i \leq r$ or $< r$, $x_i \geq b_i$	shift, add a slack, stars and bars	"inequality", "$x_i \geq -2$"

2 Mistakes, "spot the error" and a checklist

Mistakes that cost marks

1. **Forgetting the order on the shelves.** "Each book chooses a shelf" only decides which books go where; the books on each shelf can still be permuted.
2. **Treating a round table like a row.** Seating the first group in $n!$ ways counts rotations as different; using $m+1$ gaps instead of m lets two people meet in the "end" gap.
3. **Ordered versus unordered ranks.** Two ranks with the same role chosen as $13 \cdot 12$ count every hand twice; two ranks with different roles chosen as $\binom{13}{2}$ count only half of the hands.
4. **Hands that secretly upgrade.** Choosing the remaining cards freely can create an extra pair or triple. Name the exact pattern and make the extra cards obey it.
5. **Losing the constant term or the numbers in a multinomial.** Exponents add up to the power of the bracket; the constant is a block with its own exponent; every number is raised to its block's exponent, sign included.
6. **Using a binomial identity without naming it.** Write "by the binomial theorem with $x = \ldots$".
7. **Strict inequality with a slack ≥ 0.** That also counts the solutions with sum exactly r.
8. **Shifting a negative bound the wrong way.** For $x_i \geq -b$, put $y_i = x_i + b$; the right-hand side *increases*.

Spot the error 1: books on three shelves

"*Solution.*" Eight different books go on three shelves, every shelf non-empty. Each book chooses one of the three shelves: 3^8 ways. Remove the placements with an empty shelf by inclusion-exclusion: $3^8 - 3 \cdot 2^8 + 3 = 5796$.

Spot the error 2: adults and children at a round table

"*Solution.*" Seven adults and four children sit at a round table, no two children next to each other. Seat the adults: 6! ways. Write them as $_\,A\,_\,A\,_ \cdots _\,A\,_$: there are 8 gaps. Choose 4 gaps for the children and arrange them: $\binom{8}{4}\,4!$. Answer: $6!\binom{8}{4}4!$.

Spot the error 3: two triples in a six-card hand

"*Solution.*" Choose the rank of the first triple (13) and its suits ($\binom{4}{3}$), then the rank of the second triple (12) and its suits ($\binom{4}{3}$). Answer: $13 \cdot 4 \cdot 12 \cdot 4 = 2496$.

Spot the error 4: a coefficient

"*Solution.*" The coefficient of x^2y^3 in $(x-2y+3)^7$: the monomial has degree 5, so it is $\binom{5}{2,3} \cdot 1^2(-2)^3 = 10 \cdot (-8) = -80$.

Spot the error 5: lower bounds

"*Solution.*" Count the integer solutions of $x_1 + x_2 + x_3 + x_4 < 20$ with $x_1 \geq -2$, $x_2 \geq 1$, $x_3, x_4 \geq 0$. Put $y_1 = x_1 - 2$ and $y_2 = x_2 - 1$, so that $y_1, y_2 \geq 0$. Then $\sum y_i = \sum x_i - 3 < 17$, i.e. $\sum y_i \leq 16$. With a slack: $\binom{5+16-1}{16} = \binom{20}{4}$.

Checklist

- ☐ I stated what is counted and what counts as "the same": order on a shelf? rotations at a table? a hand as a set?
- ☐ For shelves, I used **row** + **cuts** and checked whether empty shelves are allowed.
- ☐ For a round table, I seated one group in $(m-1)!$ ways and used m **gaps**, not $m+1$.
- ☐ For card hands, I wrote the **rank pattern**, chose ranks **in order** for different roles and **as a set** for equal roles, and made sure the hand cannot upgrade.
- ☐ For "at least" or "weight" conditions, I listed the **exact compositions** and explained why the list is complete.
- ☐ For a coefficient, my exponents add up to the **power of the bracket**, and I included the constant and every sign.
- ☐ For a sum, I named the **binomial theorem** and the values used, and checked $n = 2$.
- ☐ For an inequality, I converted "$<$" correctly, **shifted** the lower bounds, and added a **slack variable**.

3 Exercises

Write each solution in full: statement, set-up, a step-by-step count naming the rule used at each step, and a boxed answer.

E1. A reader has 12 different books and three shelves. In how many ways can the books be placed (a) so that every shelf has at least one book; (b) if shelves may be empty?

E2. (a) Six men and six women sit at a round table so that men and women alternate. How many seatings are there? (b) Eight women and five men sit on a straight bench of 13 seats so that no two men are next to each other. How many seatings are there?

E3. (a) How many five-card hands have five different ranks? (b) How many five-card hands contain at least one ace? (c) How many six-card hands have the pattern $4 + 2$?

E4. For the ternary strings of length 8: (a) how many have weight 5? (b) how many have at most two 0's?

E5. (a) Find the coefficient of x^3y^2z in $(x + 2y - z + 1)^8$. (b) What is the sum of *all* the coefficients in the expansion of $(x - 2y + 3z + 1)^9$? (Hint: which values of x, y, z turn the expansion into the sum of its coefficients?)

E6. Evaluate, for every positive integer n: (a) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} 2^i$; (b) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} (-1)^i 4^{n-i}$.

E7. How many integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 \leq 30$ have with $x_1 \geq 5$ and $x_i \geq 0$ for $2 \leq i \leq 5$?

E8. How many integer solutions does $x_1 + x_2 + x_3 \leq 15$ have with $x_1 \geq -1$, $x_2 \geq 2$ and $x_3 \geq 0$?

E9. Use dividers to explain the formula $\frac{(n+k-1)!}{(k-1)!}$ for n distinct books on k shelves when shelves may be empty, and check it by listing for $n = 2$, $k = 2$.

E10. How many integer solutions does $x_1 + x_2 + x_3 + x_4 = 30$ have with x_1, x_2 *positive* and $x_3, x_4 \geq 3$?

A Answers to the "spot the error" puzzles

Spot the error 1 (no order on the shelves). The count $3^8 - 3 \cdot 2^8 + 3$ is the number of ways to decide *which* books go on each shelf. It ignores the order of the books on each shelf. Row and cuts give the right count: $8!\binom{7}{2} = 40\,320 \cdot 21 = 846\,720$.

Spot the error 2 (row gaps on a circle). Around a table the 7 adults create only 7 gaps. The picture $_\, A \cdots A \,_$ is a row: its first and last gaps are the same place at the table. Choosing both puts two children next to each other. Correct: $6!\binom{7}{4}4! = 720 \cdot 35 \cdot 24 = 604\,800$.

Spot the error 3 (ordered ranks for the same role). "First triple" and "second triple" are not different roles: every hand is produced twice, once with each rank first. Correct: $\binom{13}{2}\binom{4}{3}^2 = 78 \cdot 16 = 1248 = 2496/2$.

Spot the error 4 (missing constant). The bracket has power 7, so 7 blocks are chosen; the other 2 are the constant 3. Correct: $\binom{7}{2,3,2} \cdot 1^2(-2)^3 3^2 = 210 \cdot (-72) = -15\,120$.

Spot the error 5 (shift in the wrong direction). From $x_1 \geq -2$ we need $y_1 = x_1 + 2$ (the shift $y_2 = x_2 - 1$ is right). Then $\sum y_i = \sum x_i + 1 < 21$, i.e. ≤ 20, and the answer is $\binom{5+20-1}{20} = \binom{24}{4} = 10\,626$. The "solution" describes $x_1 \geq 2$, which allows *fewer* solutions; it gives only $\binom{20}{4} = 4845$.