Tutorial 4: Positive Solutions, Systems of Equations, Induction and Fibonacci Numbers

Positive solutions, the symmetry of $\binom{N}{a}$, systems of equations, induction for bounds, from two terms to n terms, and Fibonacci identities

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count **positive** integer solutions of $x_1 + \cdots + x_k = n$ with $\binom{n-1}{k-1}$, and use the **symmetry** $\binom{N}{a} = \binom{N}{N-a}$ to decide when two binomial coefficients are equal.
2. Count solutions of a **system** of equations by splitting the variables into independent groups and multiplying.
3. Prove an **upper bound** or an exact count by induction when the step splits a problem into two halves, and keep inequalities pointing the right way.
4. **Grow** a statement about two objects into a statement about n objects, using grouping and the inductive hypothesis.
5. Prove identities about the **Fibonacci numbers** by induction, using the recurrence at exactly the right step.

Sections 1.1 to 1.5 review the tools with the intuition, a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 collects the classic mistakes, "spot the error" puzzles and a checklist, and Section 3 has practice exercises; the puzzles are answered in the appendix, so the notes work for **self-study**. Tutorials 1 to 3 (counting rules, $\binom{n}{r}$, stars and bars, slack variables, proof by induction) are assumed.

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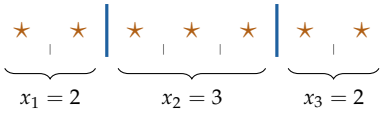
1 The toolbox

Two flavours of question meet in this tutorial: **counting** questions that reuse stars and bars and the rule of product in a new way, and **proofs by induction**. Every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap.

1.1 Positive solutions and the symmetry of binomial coefficients

The intuition

Stars and bars (Tutorial 2) counts *non-negative* solutions: a variable may be 0. If every variable must be **at least** 1, give each variable one star first. What is left is an ordinary non-negative problem. There is also a direct picture: line up the n stars and put the $k-1$ bars **in the gaps between stars**, at most one bar per gap, so that no part is empty.
The second idea of this section: $\binom{N}{a}$ counts the ways to *choose* a objects, which is the same as choosing the $N-a$ objects to *leave out*. So $\binom{N}{a} = \binom{N}{N-a}$, and two different lower numbers can give the same binomial coefficient.



7 stars, 6 inner gaps,
choose 2 gaps for the bars:
$\binom{6}{2} = 15$ positive solutions

Figure 1. The positive solution $(2,3,2)$ of $x_1 + x_2 + x_3 = 7$. The bars may only go in the $n-1 = 6$ gaps between stars (the small ticks), never at an end and never two in the same gap, so every part gets at least one star.

Simple example: $x_1 + x_2 + x_3 = 5$ with all $x_i \geq 1$

Shift. Put $y_i = x_i - 1 \geq 0$. Then $y_1 + y_2 + y_3 = 2$, which has $\binom{3+2-1}{2} = \binom{4}{2} = 6$ non-negative solutions.
Gaps. 5 stars have 4 inner gaps; choose 2 of them for the bars: $\binom{4}{2} = 6$.
Check by listing. $(3,1,1), (1,3,1), (1,1,3), (2,2,1), (2,1,2), (1,2,2)$: six solutions. ✓

Positive solutions and symmetry

(i) **Positive solutions.** Let $k \geq 1$. The number of solutions of $x_1 + \cdots + x_k = n$ in *positive* integers is

$$\binom{n-1}{k-1} \quad \text{if } n \geq k, \qquad \text{and } 0 \text{ if } n < k.$$

Equivalently, after the shift $y_i = x_i - 1$: $\binom{k+(n-k)-1}{n-k} = \binom{n-1}{n-k}$, the same number.

(ii) **Symmetry.** For integers $0 \leq a \leq N$: $\binom{N}{a} = \binom{N}{N-a}$.

(iii) **When are two binomials equal?** For integers $0 \leq a, b \leq N$:

$$\binom{N}{a} = \binom{N}{b} \iff a = b \text{ or } a + b = N.$$

(In row N of Pascal's triangle the entries increase up to the middle and then decrease, and each value appears at most twice, in mirror positions.)

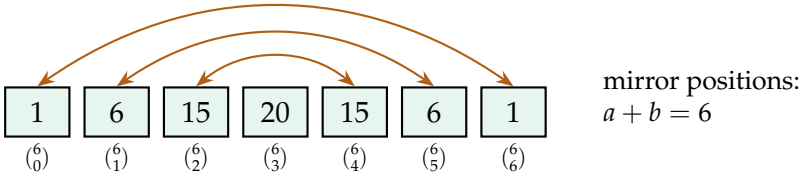


Figure 2. Row 6 of Pascal's triangle. Equal values sit in mirror positions a and $6 - a$; apart from that, all values are different.

Comprehensive example: same number of positive solutions

For which positive integer n do $x_1 + \cdots + x_5 = n$ and $y_1 + \cdots + y_{12} = n$ have the same (non-zero) number of positive integer solutions?
Solution. By (i), the counts are $\binom{n-1}{4}$ and $\binom{n-1}{11}$ (for $n \geq 12$ both are non-zero). By (iii) they are equal exactly when $4 = 11$ (impossible) or $4 + 11 = n - 1$, that is, $n = 16$.
Check. $\binom{15}{4} = 1365 = \binom{15}{11}$. ✓

Two slips with positive solutions

- Using the non-negative formula $\binom{n+k-1}{n}$ for a question that says *positive*. "Positive" means ≥ 1; "non-negative" means ≥ 0.
- Concluding "$\binom{N}{a} = \binom{N}{b}$ forces $a = b$". Equal binomials also come from mirror positions $a + b = N$; forgetting this makes it look as if there is no answer.

Positive solutions in one breath

- $x_1 + \cdots + x_k = n$, all $x_i \geq 1$: $\binom{n-1}{k-1}$ (bars in the $n - 1$ inner gaps).
- $\binom{N}{a} = \binom{N}{b}$ iff $a = b$ or $a + b = N$.

1.2 Systems of equations: split into independent parts

The intuition

Two equations that share variables look harder than one. But if one equation contains *only some* of the variables of the other, subtract: $x_1 + \cdots + x_6 = 25$ and $x_1 + x_2 = 7$ together say exactly the same as $x_1 + x_2 = 7$ and $x_3 + \cdots + x_6 = 18$. Now the two equations have **no variable in common**, so a solution of the system is **a solution of the first part together with a solution of the second part**: rule of product.

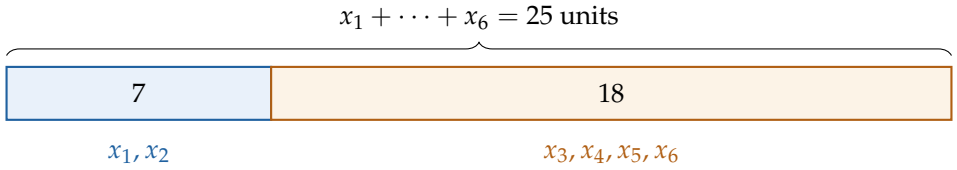


Figure 3. The second equation reserves 7 of the 25 units for x_1, x_2. The other 18 units go to $x_3, \ldots, x_6$, independently of how the first 7 were shared.

Simple example: a tiny system

Count non-negative solutions of $x_1 + x_2 + x_3 + x_4 = 5$ and $x_1 + x_2 = 2$. Subtracting gives $x_3 + x_4 = 3$. The first part $x_1 + x_2 = 2$ has 3 solutions, the second part $x_3 + x_4 = 3$ has 4. By the rule of product: $3 \cdot 4 = 12$.
Check by listing the first part. For each of $(2,0), (1,1), (0,2)$ the second part can be $(3,0), (2,1), (1,2), (0,3)$: $3 \cdot 4 = 12$. ✓

Independent parts

Suppose the variables split into disjoint groups $G_1, \ldots, G_t$, and the system is equivalent to one equation per group: $\sum_{x \in G_j} x = r_j$ for $j = 1, \ldots, t$, together with conditions that each involve only one group. Then

$$\#\text{solutions of the system} = \prod_{j=1}^{t} \#\text{solutions of the equation for } G_j.$$

The equivalence must go *both ways*: every solution of the system satisfies the new equations, and every choice of solutions for the groups satisfies the original system. Subtracting one equation from another and adding it back gives exactly this.

Comprehensive example: a three-part system

How many non-negative integer solutions have $x_1 + \cdots + x_6 = 20$, $x_1 + x_2 = 5$ and $x_3 + x_4 = 8$, if moreover $x_5 \geq 2$?
Solution. Subtracting, the system is equivalent to $x_1 + x_2 = 5$, $x_3 + x_4 = 8$, $x_5 + x_6 = 7$, three equations on disjoint groups. The condition $x_5 \geq 2$ only involves the third group: put $y_5 = x_5 - 2$, so $y_5 + x_6 = 5$. The counts are $\binom{6}{1} = 6$, $\binom{9}{1} = 9$ and $\binom{6}{1} = 6$. By the rule of product: $6 \cdot 9 \cdot 6 = 324$.

Product, not sum

The two parts are chosen *together* (one solution for the first part **and** one for the second), so the counts are multiplied. Adding them would count "a solution of the first part **or** a solution of the second part", which is not a solution of the system at all.

Systems in one breath

- Subtract to make the groups of variables disjoint; then count each group separately and multiply.
- Extra conditions (> 0, ≥ 2) are handled inside the group they belong to.

1.3 Induction for divide-and-conquer: split in half

The intuition

Some problems of size 2^{k+1} are solved by solving **two problems of size** 2^k and then **combining** the answers. A knockout tournament is the classic picture: to find the champion of 2^{k+1} players, find the champion of each half, then play one final. Induction on k fits this perfectly: the inductive hypothesis tells us how much the two halves cost, and we only have to add the cost of

the combining step.
When we prove an **upper bound** ("at most"), we may use $\leq$ at every step, and at the end it is enough to land *at or below* the target.

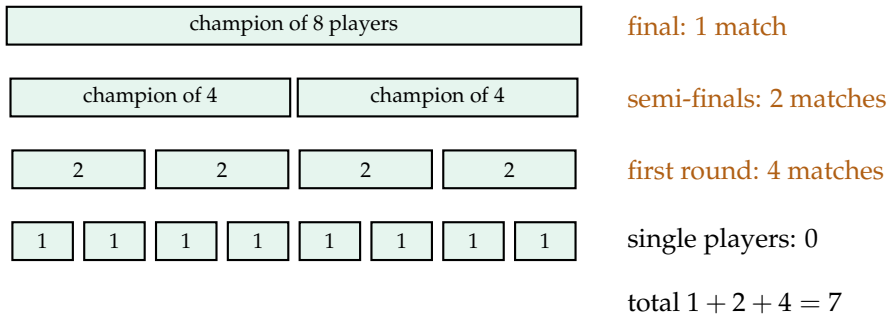


Figure 4. A knockout tournament on $2^3 = 8$ players, read from the bottom. Each half is a smaller tournament; one final combines them. Induction turns this picture into a proof.

Simple example: four players

Four players A, B, C, D. Split into $\{A, B\}$ and $\{C, D\}$: each half needs 1 match to find its champion. The two champions then play 1 final. Total $1 + 1 + 1 = 3$ matches.

Proving a statement about $T(2^n)$ by induction

Let $T(m)$ be the cost of solving an input of size m. To prove a statement about $T(2^n)$ for all $n \geq 0$:
(i) **Base case.** Check $n = 0$, that is, $T(2^0) = T(1)$.
(ii) **Inductive hypothesis.** For an arbitrary $k \geq 0$, assume the statement for *every* input of size 2^k.
(iii) **Inductive step.** Split an input of size 2^{k+1} into two inputs of size 2^k, apply the hypothesis to *each* half, add the cost of combining, and compare the total with the claim for $k+1$ ("$=$" for an exact count, "$\leq$" for a bound).

Comprehensive example: an upper bound

Suppose $T(1) = 1$ and $T(2^{k+1}) \leq 2\,T(2^k) + 3$ for all $k \geq 0$ (two halves, plus at most 3 steps to combine). Prove that $T(2^n) \leq 4 \cdot 2^n - 3$ for all $n \geq 0$.
Solution. *Base case.* $T(1) = 1$ and $4 \cdot 2^0 - 3 = 1$, so $T(1) \leq 1$. *Inductive step.* Let $k \geq 0$ and assume $T(2^k) \leq 4 \cdot 2^k - 3$. Then

$$T(2^{k+1}) \;\leq\; 2\,T(2^k) + 3 \;\overset{\text{IH}}{\leq}\; 2\big(4 \cdot 2^k - 3\big) + 3 \;=\; 4 \cdot 2^{k+1} - 3,$$

which is the claim for $n = k + 1$. By induction it holds for all $n \geq 0$.
Check. With equality at every step: $T(1) = 1$, $T(2) = 5$, $T(4) = 13$, $T(8) = 29 = 4 \cdot 8 - 3$. ✓

Three slips in divide-and-conquer inductions

- **Base case at the wrong place.** If the claim is for all $n \geq 0$, the base case is $n = 0$, not $n = 1$.
- **A hypothesis that is too weak.** The IH must be about *every* input of size 2^k, because it is applied to two different halves.
- **Starting from the goal.** Writing the inequality you want and simplifying it to "$0 \leq 1$" only

proves something if every step can be reversed. Safer: start from the cost and push it *up* to the bound.

Divide-and-conquer induction in one breath

- Split, apply the IH to each half, add the combining cost, then compare with the claim for $k+1$.
- Base case $n = 0$ when the claim starts at $n = 0$.

1.4 From two objects to n objects

The intuition

Many facts are first proved for **two** objects: $|xy| = |x||y|$, "the union of two finite sets is finite", "the sum of two even numbers is even". To extend them to n objects, **group** the first k objects into one block: $x_1 x_2 \cdots x_k x_{k+1} = (x_1 \cdots x_k)\, x_{k+1}$. Now it is a two-object situation again. Use the two-object fact once, and the inductive hypothesis takes care of the block.

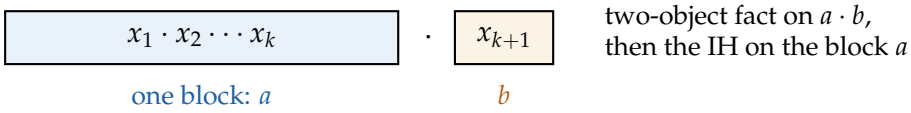


Figure 5. Grouping turns $k+1$ objects into two. The two-object fact is used with $a = x_1 \cdots x_k$ and $b = x_{k+1}$.

Simple example: three even numbers

From "the sum of two even integers is even", show that $a + b + c$ is even when a, b, c are even. Group: $a + b + c = (a + b) + c$. The two-number fact applied to a, b shows $a + b$ is even; applied again to the even numbers $a + b$ and c, it shows $(a + b) + c$ is even. Induction does exactly this, once for each new number.

Extending a two-object property by induction

Suppose a property holds for any two objects: $\Phi(a, b)$ for all a, b. To prove the n-object version for all $n \geq 2$:

(i) **Base case** $n = 2$: this is the two-object fact.
(ii) **Inductive hypothesis**: for an arbitrary $k \geq 2$, the statement holds for *any* k objects.
(iii) **Inductive step**: given $k+1$ objects, group the first k (associativity), apply the two-object fact to the block and the last object, then apply the IH to the block.

Comprehensive example: absolute value of a product

Given $|ab| = |a|\,|b|$ for all real a, b, prove $|x_1 x_2 \cdots x_n| = |x_1|\,|x_2| \cdots |x_n|$ for all $n \geq 2$ and all reals $x_1, \ldots, x_n$.

Solution. *Base case* $n = 2$: given. *Inductive step.* Let $k \geq 2$ and assume the statement for any k reals. For reals $x_1, \ldots, x_{k+1}$:

$$|x_1 \cdots x_k x_{k+1}| = |(x_1 \cdots x_k)\, x_{k+1}| = |x_1 \cdots x_k|\,|x_{k+1}| \overset{\text{IH}}{=} |x_1| \cdots |x_k|\,|x_{k+1}|.$$

The first equality is associativity, the second the two-term fact with $a = x_1 \cdots x_k$, $b = x_{k+1}$. By

induction the statement holds for all $n \geq 2$.

Where the two-object fact is used

In the example, $|ab| = |a||b|$ is applied with $a = x_1 \cdots x_k$, which is *one real number*. Say so. A step that jumps from $|x_1 \cdots x_{k+1}|$ straight to $|x_1| \cdots |x_{k+1}|$ "by the IH" uses the statement for $k+1$ objects, which is what we are trying to prove.

Two objects to n objects in one breath

- Group the first k objects, use the two-object fact once, then the IH on the group.
- Base case $n = 2$; name both justifications in the step.

1.5 Fibonacci numbers and recursive definitions

The intuition

The Fibonacci numbers are defined by **a starting point and a rule**: $F_0 = 0$, $F_1 = 1$, and each later number is the sum of the two before it. Proofs about them by induction have one special ingredient: somewhere in the inductive step you must **use the recurrence** $F_m = F_{m-1} + F_{m-2}$, and you should say so. For sums of Fibonacci numbers, the step always starts the same way: **split off the last term**.

n	0	1	2	3	4	5	6	7	8
F_n	0	1	1	2	3	5	8	13	21

Table 1. The first Fibonacci numbers.

Simple example: computing with the recurrence

$F_2 = F_1 + F_0 = 1$, $F_3 = F_2 + F_1 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$. Notice that $F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6$: a pattern worth proving (see the comprehensive example).

Fibonacci numbers

The Fibonacci numbers are defined recursively by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Splitting off the last term. For a sum of $k+1$ terms $b_1, \ldots, b_{k+1}$: $\sum_{i=1}^{k+1} b_i = \left(\sum_{i=1}^{k} b_i\right) + b_{k+1}$. Identify the new last term carefully.

Using the recurrence. Any two consecutive Fibonacci numbers add up to the next one, for instance $F_{2k} + F_{2k+1} = F_{2k+2}$ (the recurrence with $n = 2k+2 \geq 2$).

Comprehensive example: sum of the odd-indexed terms

Prove that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ for all $n \geq 1$.
Solution. *Base case* $n = 1$. The left side is $F_1 = 1$ and the right side is $F_2 = 1$. *Inductive step.* Let $k \geq 1$ and assume $F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}$. The sum for $n = k+1$ has one more term, F_{2k+1}:

$$F_1 + \cdots + F_{2k-1} + F_{2k+1} \overset{\text{IH}}{=} F_{2k} + F_{2k+1} = F_{2k+2} = F_{2(k+1)},$$

by the recurrence. By the principle of mathematical induction the identity holds for all $n \geq 1$.
Check ($n = 3$): $F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6$. ✓

Two slips with Fibonacci induction

- **The wrong last term.** In $F_1 + F_3 + \cdots + F_{2n-1}$, going from $n = k$ to $n = k+1$ adds the term $F_{2(k+1)-1} = F_{2k+1}$, not F_{2k+3}. Write the new sum out with its last term before you simplify.
- **A different starting convention.** Some books start with $F_0 = 1$, $F_1 = 1$. State the definition you use. The course uses $F_0 = 0$, $F_1 = 1$.

Fibonacci induction in one breath

- Write the sum for $k+1$ as the sum for k plus its last term, use the IH, then the recurrence.
- Name the recurrence when you use it: "since $F_{2k+2} = F_{2k+1} + F_{2k}$".

The toolbox at a glance

Situation	Count / method	Signal words
$x_1 + \cdots + x_k = n$, all $x_i \geq 1$	$\binom{n-1}{k-1}$	"positive integer solutions"
Two binomials are equal	$a = b$ or $a + b = N$	"for which n … the same number"
Equations sharing variables	subtract, split, multiply	"the pair of equations"
Divide and conquer	induction on n, IH on each half	"size 2^n", "two halves"
Two-object fact to n objects	group, two-object fact, then IH	"for $n \geq 2$ and $x_1, \ldots, x_n$"
Identity with F_n	split off last term, IH, recurrence	"Fibonacci", "prove by induction"

2 Mistakes, "spot the error" and a checklist

Mistakes that cost marks

1. **Non-negative instead of positive.** $\binom{n+k-1}{n}$ counts non-negative solutions; for positive variables use $\binom{n-1}{k-1}$.
2. **Forgetting the mirror solution.** $\binom{N}{a} = \binom{N}{b}$ does not need $a = b$; $a + b = N$ also works.
3. **Adding instead of multiplying** the counts of independent parts of a system.
4. **Base case at $n = 1$ for a claim about $n \geq 0$.**
5. **Using the claim for $k+1$ objects inside its own proof.** Group first, then the two-object fact, then the IH.

6. **Adding the wrong last term** in a sum of Fibonacci numbers.

Spot the error 1

"Solution." The equations $x_1 + \cdots + x_5 = n$ and $y_1 + \cdots + y_{12} = n$ have $\binom{n-1}{4}$ and $\binom{n-1}{11}$ positive solutions. For these to be equal we need $4 = 11$, which is impossible. So no such n exists.

Spot the error 2

"Solution." The system $x_1 + x_2 + x_3 + x_4 = 5$, $x_1 + x_2 = 2$ splits into $x_1 + x_2 = 2$ (3 solutions) and $x_3 + x_4 = 3$ (4 solutions). By the rule of sum the system has $3 + 4 = 7$ solutions.

Spot the error 3

"Proof that a tournament with 2^n players needs $2^n - 1$ matches, for all $n \geq 0$." Base case $n = 1$: two players play $1 = 2^1 - 1$ match. Inductive step: two halves of 2^k players need $2^k - 1$ matches each, plus the final: $2(2^k - 1) + 1 = 2^{k+1} - 1$. Hence the formula holds for all $n \geq 0$.

Spot the error 4

"Inductive step." $F_1 + \cdots + F_{2k-1} + F_{2k+3} \overset{\text{IH}}{=} F_{2k} + F_{2k+3}$, and since the target is F_{2k+2}, the identity fails, so the claim is false.

Spot the error 5

"Inductive step." Let $x_1, \ldots, x_{k+1}$ be reals. By the inductive hypothesis, $|x_1 x_2 \cdots x_{k+1}| = |x_1||x_2| \cdots |x_{k+1}|$. This proves the statement for $k + 1$.

Checklist

- ☐ I checked whether the variables are **positive** or **non-negative**, and shifted accordingly.
- ☐ When two binomial coefficients must be equal, I considered both $a = b$ and $a + b = N$.
- ☐ For a system, I **subtracted** to get disjoint groups, said why the new system is equivalent, and **multiplied**.
- ☐ In every induction proof, the **base case** matches the start of the claim.
- ☐ My inductive hypothesis is written out in full, and is about **every** input / **any** k objects.
- ☐ In the step, I wrote the goal $P(k+1)$ before computing, and **marked** where the IH is used.
- ☐ For an upper bound, my chain of $\leq$ ends **at or below** the target.
- ☐ For Fibonacci, I split off the **correct last term** and named the recurrence.

3 Exercises

Write each solution in full, using the counting or induction template from Tutorials 1 and 2.

E1. Find all a with $0 \leq a \leq 20$ such that $\binom{20}{a} = \binom{20}{7}$, and explain why there are no others.

E2. How many non-negative integer solutions have $x_1 + \cdots + x_6 = 20$, $x_1 + x_2 = 5$ and $x_3 + x_4 = 8$?

E3. How many non-negative integer solutions of $x_1 + \cdots + x_6 = 25$, $x_1 + x_2 = 7$ have $x_i \geq 2$ for $3 \leq i \leq 6$?

E4. Prove by induction that a knockout tournament with 2^n players needs exactly $2^n - 1$ matches, for all $n \geq 0$. (Then find a one-line proof without induction: every match eliminates exactly one player.)

E5. Using $|ab| = |a||b|$, prove that $|x_1 x_2 \cdots x_n| = |x_1||x_2| \cdots |x_n|$ for all $n \geq 2$ and reals $x_1, \ldots, x_n$.

E6. Prove by induction, for $n \geq 0$: (a) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$; (b) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for $n \geq 1$ (Cassini's identity).

E7. Prove by induction that $F_n \leq 2^n$ for all $n \geq 0$. (You will need two base cases. Why?)

E8. Using "the union of two finite sets is finite", prove that the union of n finite sets is finite for every $n \geq 2$.

A Answers to the "spot the error" puzzles

Spot the error 1 (forgetting symmetry). Equal lower numbers are one way for two binomial coefficients with the same upper number to be equal, but not the only one: $\binom{N}{a} = \binom{N}{N-a}$. Here the counts are equal when $4 + 11 = n - 1$, i.e. $n = 16$.

Spot the error 2 (sum instead of product). A solution of the system needs *both* a solution of $x_1 + x_2 = 2$ *and* a solution of $x_3 + x_4 = 3$, chosen independently: $3 \cdot 4 = 12$.

Spot the error 3 (base case). The claim is for all $n \geq 0$, but the proof starts at $n = 1$, so $n = 0$ is never covered. Check $n = 0$ (one player, $0 = 2^0 - 1$ matches) instead, or as well.

Spot the error 4 (wrong last term). The sum for $n = k + 1$ ends with $F_{2(k+1)-1} = F_{2k+1}$, not F_{2k+3}. With the correct term, $F_{2k} + F_{2k+1} = F_{2k+2}$ and the step works. A wrong step proves nothing about the claim, true or false.

Spot the error 5 (using what we want to prove). The inductive hypothesis is about k reals, but it is applied to $k + 1$ reals. Group first: $|(x_1 \cdots x_k)x_{k+1}| = |x_1 \cdots x_k|\,|x_{k+1}|$ by the two-term fact, and only then use the IH on $|x_1 \cdots x_k|$.