Tutorial 5: Inclusion and Exclusion, Derangements and First Generating Functions

Upper bounds, "exactly" and "at least", gluing letters, derangements, generating functions for distributions, and palindromes of integers

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Count integer solutions with **upper bounds** by turning each bound into a "bad" condition and using the **principle of inclusion and exclusion** (PIE).
2. **Prepare** an equation before counting: shift lower bounds to 0 and **rescale** when every value is a multiple of the same number.
3. Count the objects that satisfy **exactly** m or **at least** m of the conditions (E_m and L_m).
4. Count arrangements with **identical letters next to each other** by gluing them into a block.
5. Count permutations with a given number of **fixed points** using **derangements** d_n.
6. Write the **generating function** of a distribution problem: one factor per recipient, one power of x per allowed amount.
7. Count **compositions** and **palindromes of an integer**.

Sections 1.1 to 1.7 review the tools with the intuition, a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 collects the classic mistakes, "spot the error" puzzles and a checklist, and Section 3 has practice exercises; the puzzles are answered in the appendix, so the notes work for **self-study**. Tutorials 1 to 4 (rules of sum and product, $\binom{n}{r}$, permutations with repetition, stars and bars, shifting lower bounds) are assumed.

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1 The toolbox

Most of this tutorial is about one big idea: **count everything, then remove what is bad, then fix what you removed twice**. The last two sections start the next chapter: generating functions. Every tool is explained the same way: the idea in plain words, a small example, the precise statement, a bigger example, and a recap.

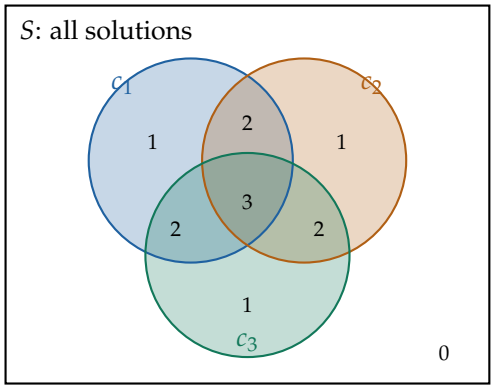
1.1 The principle of inclusion and exclusion

The intuition

Stars and bars (Tutorial 2) counts solutions with *lower* bounds easily, but *upper* bounds such as $x_i \leq 6$ are hard to build in directly. The trick is to flip the question. Call a solution **bad for** x_i if it breaks the bound, that is, $x_i \geq 7$. Bad solutions *are* easy to count: give x_i its 7 units first, then use stars and bars. So:

$$\#\text{good} = \#\text{all} - \#\text{bad}.$$

The catch: a solution can be bad for two variables at once, and then subtracting "bad for x_1" and "bad for x_2" removes it **twice**. Add those back once. But a solution bad for three variables was removed three times and added back three times, so remove it once more, and so on. The signs alternate: $-, +, -, +, \ldots$



numbers: how many of the conditions the region satisfies

$S_0 - S_1 + S_2 - S_3$ counts a point in a region marked r exactly $\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3}$ times, which is 1 if $r = 0$ and 0 if $r \geq 1$.

Figure 1. Three conditions c_1, c_2, c_3 ("bad" properties). We want the region marked 0: the solutions that satisfy none of them. Inclusion and exclusion is a bookkeeping rule that counts that region exactly once and every other region zero times.

Simple example: $x_1 + x_2 + x_3 = 6$ with $0 \leq x_i \leq 3$

All non-negative solutions: $\binom{3+6-1}{6} = \binom{8}{2} = 28$.
Bad for x_1 ($x_1 \geq 4$): put $y_1 = x_1 - 4 \geq 0$, so $y_1 + x_2 + x_3 = 2$: $\binom{4}{2} = 6$ solutions. The same for x_2 and x_3, so $S_1 = 3 \cdot 6 = 18$.
Bad for two variables ($x_1 \geq 4$ and $x_2 \geq 4$): the sum would be at least $8 > 6$, impossible. So $S_2 = 0$.
Good: $28 - 18 + 0 = 10$.
Check by listing. The permutations of $(3,3,0)$: 3; of $(3,2,1)$: 6; and $(2,2,2)$: 1. Total 10. ✓

Principle of inclusion and exclusion

Let S be a finite set with $|S| = N$, and let $c_1, \ldots, c_t$ be conditions that an element of S may or may not satisfy. Write $N(c_i c_j)$ for the number of elements that satisfy c_i **and** c_j (and maybe others), and $N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_t})$ for the number that satisfy **none** of them. Put

$$S_0 = N, \qquad S_1 = \sum_i N(c_i), \qquad S_2 = \sum_{i<j} N(c_i c_j), \qquad \ldots, \qquad S_t = N(c_1 c_2 \cdots c_t).$$

Then

$$N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_t}) = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

Symmetry shortcut. If every choice of k conditions gives the same number, then $S_k = \binom{t}{k} \cdot N(c_1 c_2 \cdots c_k)$.

Comprehensive example: different upper bounds

Count integer solutions of $x_1 + x_2 + x_3 = 12$ with $0 \le x_1 \le 4$, $0 \le x_2 \le 5$, $0 \le x_3 \le 6$.
Solution. Conditions: $c_1 : x_1 \ge 5$, $c_2 : x_2 \ge 6$, $c_3 : x_3 \ge 7$. In each case remove the forced units and count non-negative solutions of what is left (with 3 variables, $\binom{r+2}{2}$ solutions for a right-hand side r):

Term & conditions	units removed	new right-hand side	count
S_0	0	12	$\binom{14}{2} = 91$
$N(c_1), N(c_2), N(c_3)$	5, 6, 7	7, 6, 5	36, 28, 21
$N(c_1c_2), N(c_1c_3), N(c_2c_3)$	11, 12, 13	1, 0, -1	3, 1, 0
$N(c_1c_2c_3)$	18	-6	0

So $N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_3}) = 91 - (36 + 28 + 21) + (3 + 1 + 0) - 0 = 91 - 85 + 4 = 10$.

Two slips with upper bounds

- **The bad condition is one more than the bound.** "$x_i < 7$" is broken by $x_i \ge 7$, but "$x_i \le 7$" is broken by $x_i \ge 8$. Write the bad condition down explicitly for every variable.
- **A negative right-hand side means 0, not a binomial.** If the forced units exceed the total, there are no solutions. Do not plug a negative number into $\binom{r+k-1}{r}$.

PIE in one breath

- Bad condition c_i: "x_i is too big". Count with stars and bars after removing the forced units.
- Good $= S_0 - S_1 + S_2 - \cdots$; stop as soon as the S_k become 0, and *say why* they are 0.

1.2 Preparing the equation: shift, then rescale

The intuition

Before using PIE, make the problem look like "non-negative integers, possibly with upper bounds". Two moves do this.

- **Shift.** A lower bound $x_i \geq a$ is removed by giving x_i its a units first: $y_i = x_i - a \geq 0$. The right-hand side drops by a, and an upper bound $x_i \leq b$ becomes $y_i \leq b - a$.
- **Rescale.** If every value must be a multiple of 5, count in *blocks of* 5 instead of single units: $y_i = x_i/5$. Everything (the total and the bounds) is divided by 5.

After the two moves, the question is again "non-negative integers with upper bounds", and Section 1.1 takes over.

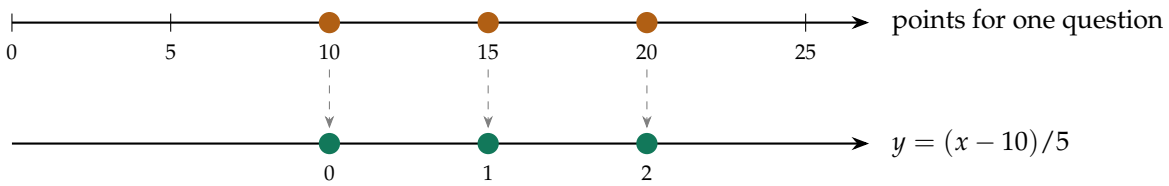


Figure 2. Shift then rescale, for the comprehensive example below: the allowed values $10, 15, 20$ *become* $0, 1, 2$. *A sum of* 4 *such values equal to* 60 *becomes a sum of* 4 *values in* $\{0, 1, 2\}$ *equal to* $(60 - 4 \cdot 10)/5 = 4$.

Simple example: coins in multiples of 5

In how many ways can 3 children share 40 cents if each gets a multiple of 5 cents and at least 5 cents?

Shift: give each child 5 cents, leaving 25 cents, each child now getting a non-negative multiple of 5. Rescale: $y_1 + y_2 + y_3 = 5$ with $y_i \geq 0$. Stars and bars: $\binom{7}{2} = 21$.

Shift and rescale

Let a, b, d be integers with $d \geq 1$ and $a \leq b$.

(i) **Shift.** The map $x_i \mapsto y_i = x_i - a$ is a bijection between integers x_i with $a \leq x_i \leq b$ and integers y_i with $0 \leq y_i \leq b - a$.

(ii) **Rescale.** If a and b are multiples of d, then $y \mapsto y/d$ is a bijection between the multiples of d in $[0, b - a]$ and the integers in $[0, (b - a)/d]$.

Applied to every variable at once, these maps turn solutions of the old equation into solutions of the new one, one to one. Shift first, then rescale: after the shift, every variable is a multiple of d that starts at 0.

Comprehensive example: shift, rescale, exclude

How many ways are there to split 60 points among 4 questions if each question is worth between 10 and 20 points and every value is a multiple of 5?

Solution. Shift by 10: $x'_1 + \cdots + x'_4 = 20$ with $0 \leq x'_i \leq 10$. Rescale by 5: $y_1 + \cdots + y_4 = 4$ with $0 \leq y_i \leq 2$. Bad condition $c_i : y_i \geq 3$. Then $S_0 = \binom{7}{3} = 35$, $S_1 = 4 \cdot \binom{4}{3} = 16$ (after removing 3, the right-hand side is 1), and $S_2 = 0$ since $3 + 3 > 4$. Answer: $35 - 16 = 19$.

Check by listing values of (y_i). The permutations of $(2, 2, 0, 0)$: 6; of $(2, 1, 1, 0)$: 12; and $(1, 1, 1, 1)$: 1. Total 19. ✓

Rescaling the bounds too

The upper bound must be shifted and rescaled exactly like the variable: with values between 5 and 20, the bound $x \leq 20$ becomes $x - 5 \leq 15$, then $y \leq 3$. A common slip is to rescale the total but keep the old bound ($y \leq 15$), which makes the PIE terms wrong.

Preparing in one breath

- Shift away the lower bounds (subtract from the total and from the upper bounds).
- If all values are multiples of d, divide the total and the bounds by d.

1.3 "Exactly m" and "at least m"

The intuition

PIE answers "how many satisfy *none* of the conditions". Often we want more: "how many satisfy **exactly one**" or "**at least one**". The same numbers S_k answer these too; only the coefficients change.

Why is a correction needed at all? S_1 counts each element once *for every* condition it satisfies. An element satisfying exactly 3 conditions is counted $\binom{3}{1} = 3$ times. To get "exactly 1", subtract those extra counts using S_2, then fix the over-correction using S_3, and so on.

An element satisfying exactly r conditions	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$
is counted in S_1 this many times: $\binom{r}{1}$	0	1	2	3	4
is counted in S_2 this many times: $\binom{r}{2}$	0	0	1	3	6
is counted in S_3 this many times: $\binom{r}{3}$	0	0	0	1	4
is counted in S_4 this many times: $\binom{r}{4}$	0	0	0	0	1
in $S_1 - 2S_2 + 3S_3 - 4S_4 = E_1$	0	**1**	0	0	0
in $S_1 - S_2 + S_3 - S_4 = L_1$	0	**1**	**1**	**1**	**1**

Table 1. Why the coefficients of E_1 and L_1 (with $t = 4$ conditions) are what they are. Every column is checked by arithmetic: for $r = 4$, $4 - 2 \cdot 6 + 3 \cdot 4 - 4 \cdot 1 = 0$ and $4 - 6 + 4 - 1 = 1$.

Simple example: digits

Among the 3-digit strings over $\{0, 1\}$, let c_i be "digit i is 1" ($i = 1, 2, 3$). Then $S_1 = 3 \cdot 4 = 12$, $S_2 = 3 \cdot 2 = 6$, $S_3 = 1$. Exactly two 1s: $E_2 = S_2 - \binom{3}{1}S_3 = 6 - 3 = 3$ (the strings 110, 101, 011). At least two: $L_2 = S_2 - \binom{2}{1}S_3 = 6 - 2 = 4$ (add 111). ✓

Generalized inclusion and exclusion

With $S_0, \ldots, S_t$ as in Section 1.1, let E_m be the number of elements of S that satisfy **exactly** m of

the conditions and L_m the number that satisfy **at least** m of them ($0 \le m \le t$). Then

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t,$$
$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t \quad (m \ge 1).$$

In words: the coefficient of S_{m+j} is $(-1)^j \binom{m+j}{j}$ in E_m and $(-1)^j \binom{m+j-1}{j}$ in L_m. For $m = 0$, E_0 is the formula of Section 1.1, and $L_0 = N$. Two checks: $L_m = E_m + E_{m+1} + \cdots + E_t$, and $E_0 + E_1 + \cdots + E_t = N$.

Comprehensive example: fixed points of a permutation of 4

How many permutations of $1, 2, 3, 4$ fix exactly one number? At least one?
Solution. Let c_i be "i is in position i". Fixing k chosen numbers leaves $(4-k)!$ arrangements, so $S_k = \binom{4}{k}(4-k)!$: $S_1 = 24$, $S_2 = 12$, $S_3 = 4$, $S_4 = 1$. Then

$$E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \binom{4}{3} S_4 = 24 - 24 + 12 - 4 = 8,$$
$$L_1 = S_1 - S_2 + S_3 - S_4 = 24 - 12 + 4 - 1 = 15.$$

Check. $L_1 = 4! - (\text{permutations with no fixed point}) = 24 - 9 = 15$. ✓

Two different coefficient rows

The coefficients of E_m are $\binom{m}{0}, \binom{m+1}{1}, \binom{m+2}{2}, \ldots$ (the *bottom* number grows), while those of L_m are $\binom{m-1}{m-1}, \binom{m}{m-1}, \binom{m+1}{m-1}, \ldots$ (the *bottom* number is fixed at $m-1$). Mixing them up is the most common error with this tool. Always check $L_m - E_m = L_{m+1}$.

Exactly and at least in one breath

- E_m: coefficients $\binom{m+j}{j}$; L_m: coefficients $\binom{m+j-1}{m-1}$; signs alternate starting with $+$.
- Sanity checks: $L_m = E_m + E_{m+1} + \cdots + E_t$, and $\sum_m E_m = N$.

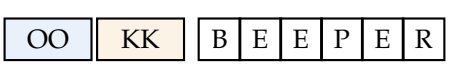
1.4 Identical letters next to each other: glue them

The intuition

"The two A's are next to each other" is a condition about an arrangement. To count arrangements that satisfy it, **glue** the two A's into one block AA and arrange the blocks. The block is one object, so the word gets **one letter shorter** for every pair that is glued. Inside the block nothing can move: the two A's are identical, so AA has only one internal order.



10 letters



OO and KK glued: 8 objects, $\frac{8!}{3!}$ arrangements

Figure 3. Gluing two pairs in BOOKKEEPER. Each glued pair removes one object. The E's are not glued here, so they are still three identical letters among the 8 objects.

Simple example: BOOK

How many arrangements of BOOK have the two O's together? Glue: $\boxed{\text{OO}}$, B, K are 3 distinct objects: $3! = 6$. Total arrangements $4!/2! = 12$, so $12 - 6 = 6$ have the O's apart. *Check by listing.* The O's apart means they occupy positions $\{1,3\}$, $\{1,4\}$ or $\{2,4\}$, and B, K fill the other two positions in 2 orders: $3 \cdot 2 = 6$. ✓

Gluing

Consider a word with n letters, and let c_i mean "the two copies of the i-th letter that appears exactly twice are adjacent". If k such doubled letters are glued, the arrangement has $n - k$ objects, the k blocks are distinct objects appearing once, and every other letter keeps its multiplicity. So

$$N(c_{i_1} \cdots c_{i_k}) = \frac{(n-k)!}{\prod(\text{multiplicities of the letters that are not glued})!}.$$

Gluing is a bijection: an arrangement in which the two A's are adjacent corresponds to exactly one arrangement of the objects with $\boxed{\text{AA}}$ as one object.

Comprehensive example: no two identical letters adjacent in AABBC

Conditions c_A, c_B. $N = \frac{5!}{2!\,2!} = 30$. Glue A's: $\boxed{\text{AA}}$, B, B, C gives $\frac{4!}{2!} = 12$; the same for B's: 12. Glue both: $\boxed{\text{AA}}$, $\boxed{\text{BB}}$, C gives $3! = 6$. So $N(\overline{c_A}\,\overline{c_B}) = 30 - 24 + 6 = 12$, and exactly one adjacent pair: $E_1 = S_1 - 2S_2 = 24 - 12 = 12$, and both: $E_2 = 6$. *Check:* $12 + 12 + 6 = 30$. ✓

Do not multiply by 2! for the inside of the block

Two *identical* A's in a block can only be "AA". Multiplying by 2! for "the order inside the block" is wrong (it would be right for two *different* people sitting together). See Spot the error 4.

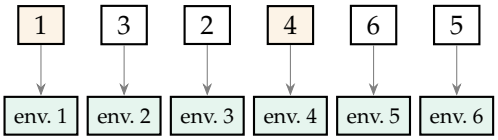
Gluing in one breath

- One glued pair = one object fewer; the block has one internal order.
- Divide by the factorials of the letters that are still repeated.

1.5 Derangements and fixed points

The intuition

A secretary types n letters and n addressed envelopes, then puts the letters into the envelopes at random. Letter i is a **fixed point** if it lands in its own envelope. A **derangement** is an arrangement with *no* fixed points: every letter is in a wrong envelope. PIE with the conditions "letter i is in envelope i" counts derangements. And to get *exactly* m fixed points there is an even shorter route: **choose** the m letters that are right, then **derange** the rest.



letters 1 and 4 are fixed points;
letters 2, 3, 5, 6 form a
derangement of the other envelopes

Figure 4. An arrangement of 6 letters with exactly 2 fixed points: the fixed letters are 1 and 4, and the other 4 letters all avoid their own envelopes.

Simple example: d_3

The permutations of $1, 2, 3$ with no fixed point are 231 and 312, so $d_3 = 2$. By the formula below: $3!\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 6 \cdot \frac{1}{3} = 2$. ✓

Derangements

The number d_n of permutations of $1, \ldots, n$ with no fixed point is ($d_0 = 1$)

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

n	0	1	2	3	4	5
d_n	1	0	1	2	9	44

Exactly m fixed points. Choose which m objects are fixed; the remaining objects must then form a derangement among themselves (rule of product).

Comprehensive example: two ways to get E_1 for $n = 5$

Choose and derange. $E_1 = \binom{5}{1} d_4 = 5 \cdot 9 = 45$.
Generalized PIE. $S_k = \binom{5}{k}(5-k)! = 5!/k!$, so $S_1 = 120$, $S_2 = 60$, $S_3 = 20$, $S_4 = 5$, $S_5 = 1$, and

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4 + 5S_5 = 120 - 120 + 60 - 20 + 5 = 45. \checkmark$$

Exactly m fixed points is not $\binom{n}{m}(n-m)!$

$\binom{n}{m}(n-m)!$ fixes m chosen letters but lets the others go anywhere, *including* their own envelopes. That is S_m, which counts each arrangement with $r \geq m$ fixed points $\binom{r}{m}$ times.

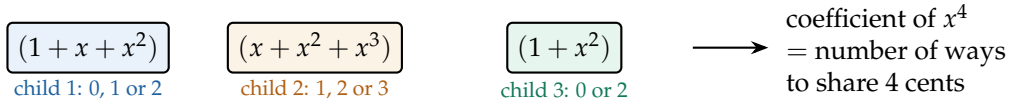
Derangements in one breath

- $d_n = n! \sum_{k=0}^{n} (-1)^k / k!$, by PIE.
- Exactly m fixed points: choose the fixed ones, derange the rest. At least m: add the exact counts, or use L_m.

1.6 Generating functions for distributions

The intuition

A generating function is a **bookkeeping device**. For each person (or variable), write a polynomial whose *exponents* are the amounts that person is allowed to receive: if a child may get $0, 1, 2, \ldots$ cents, write $1 + x + x^2 + \cdots$. Multiply the polynomials of all the people. Expanding the product means choosing one term from each factor, which is choosing one amount for each person, and the exponents **add up**. So the coefficient of x^n is exactly the number of ways to hand out n in total.



picking $x^1 \cdot x^1 \cdot x^2$ means amounts $(1, 1, 2)$; picking $x^2 \cdot x^2 \cdot x^0$ means $(2, 2, 0)$; ...

Figure 5. One factor per child, one power of x per allowed amount. Every way of choosing one term from each factor with exponents adding up to 4 is one distribution of 4 cents.

Simple example: the product of Figure 5

Expanding, $(1 + x + x^2)(x + x^2 + x^3)(1 + x^2) = x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 2x^6 + x^7$. So there are 4 ways to share 4 cents: $(2, 2, 0)$, $(1, 3, 0)$, $(1, 1, 2)$, $(0, 2, 2)$. ✓

Generating functions

The (ordinary) **generating function** of a sequence $a_0, a_1, a_2, \ldots$ is $f(x) = \sum_{n \geq 0} a_n x^n$, and $[x^n] f(x)$ denotes the coefficient a_n.

(i) **Distributions.** The number of integer solutions of $x_1 + \cdots + x_k = n$ with $x_i \in A_i$ is the coefficient of x^n in $\prod_{i=1}^{k} \left(\sum_{a \in A_i} x^a \right)$.

(ii) **Useful identities.**

$$1 + x + \cdots + x^r = \frac{1 - x^{r+1}}{1 - x}, \qquad \frac{1}{1 - x} = \sum_{n \geq 0} x^n, \qquad \frac{1}{(1 - x)^k} = \sum_{n \geq 0} \binom{k + n - 1}{n} x^n.$$

(iii) **Factoring out the minimum.** $x^a + x^{a+1} + \cdots = x^a (1 + x + \cdots) = \dfrac{x^a}{1 - x}$.

When a question asks for "the generating function", the answer is the function, not the number. Writing the coefficient you would need is a good habit.

Comprehensive example: 20 candies, 4 children

Find the generating function for distributing 20 identical candies among 4 children so that each child gets at least 2 and the first child gets at most 6, and find the count.

Solution. Child 1: $x^2 + x^3 + \cdots + x^6$. Children 2, 3, 4: $x^2 + x^3 + \cdots$ (a finite version up to x^{14} is also correct, since nobody can get more than $20 - 3 \cdot 2 = 14$). So

$$f(x) = (x^2 + \cdots + x^6)\,(x^2 + x^3 + \cdots)^3 = x^8 \frac{1 - x^5}{1 - x} \cdot \frac{1}{(1 - x)^3} = \frac{x^8 (1 - x^5)}{(1 - x)^4}.$$

The count is $[x^{20}]f = [x^{12}]\frac{1-x^5}{(1-x)^4} = \binom{15}{12} - \binom{10}{7} = 455 - 120 = 335$. Notice that the factor $1 - x^5$ does the inclusion and exclusion for us.

Two slips with generating functions

- **Shifting twice.** If you give the youngest child 3 candies first *and* write the factor $x^3 + x^4 + \cdots$, you have counted the 3 candies twice. Either shift (and use the smaller target with factor $1 + x + \cdots$) or keep $x^3 + \cdots$ with the original target. Never both.
- **Answering with the number.** "Determine the generating function" asks for $f(x)$.

Generating functions in one breath

- One factor per recipient; its exponents are the allowed amounts; the answer is $[x^n]$ of the product.
- $\frac{1}{(1-x)^k}$ has coefficients $\binom{k+n-1}{n}$; a minimum a pulls out x^a.

1.7 Compositions and palindromes of an integer

The intuition

A **composition** of n is a way to write n as an *ordered* sum of positive integers: $4 = 1 + 3$ and $4 = 3 + 1$ are different compositions. A **palindrome** of n is a composition that reads the same forwards and backwards, such as $1 + 2 + 1$ or $2 + 2$.
This is not the same as a palindrome *word*. Here the "letters" are the summands, and they must add up to n. A palindrome is decided by its **left half**: the right half is its mirror image, and whatever is left in the middle is forced.



Figure 6. Two palindromes of 6. Blue: the left half, chosen freely. Gray: its mirror image. Orange: the middle summand, forced to be whatever is left (here $6 - 2 \cdot 2 = 2$); it is absent when the two halves already add up to n.

Simple example: palindromes of 4 and 5

Palindromes of 4: $4,\ 1+2+1,\ 2+2,\ 1+1+1+1$: four of them.
Palindromes of 5: $5,\ 1+3+1,\ 2+1+2,\ 1+1+1+1+1$: also four.

Compositions and palindromes

(i) The number of compositions of $n \geq 1$ is 2^{n-1}. (Write n as n ones; each of the $n - 1$ gaps either gets a "+" or not.)
(ii) **Left halves.** A palindrome of n is decided by its left half, which is a composition of some k with $0 \leq 2k \leq n$ (the empty half for $k = 0$); the middle summand $n - 2k$ is then forced (and absent if $2k = n$). Conversely every such left half gives exactly one palindrome. So palindromes can be counted by adding, over k, the numbers of compositions of k.

Comprehensive example: the palindromes of 6 by their left half

k (left half sums to)	palindromes	count
0	6	1
1	$1+4+1$	1
2	$2+2+2,\ 1+1+2+1+1$	2
3	$3+3,\ 2+1+1+2,\ 1+2+2+1,\ 1+1+1+1+1+1$	4
	total	8

Composition, not word

A palindrome *word* is a string of letters; a palindrome *of the integer* n is a composition of n. Read which one a question means: "palindromes of 7" means compositions of the number 7 that read the same both ways.

Palindromes of integers in one breath

- Compositions of n: 2^{n-1}.
- A palindrome of n is decided by its left half; the middle is forced.

The toolbox at a glance

Situation	Count / method	Signal words
Upper bounds on variables	PIE with $c_i : x_i \geq b_i + 1$	"$x_i < 7$", "at most"
Lower bounds, multiples of d	shift, then divide by d	"at least 10", "multiple of 5"
Exactly / at least m conditions	E_m, L_m	"exactly two", "at least two"
Identical letters adjacent	glue into a block	"consecutive identical letters"
Fixed points of a permutation	choose the fixed ones, derange the rest	"in its own envelope"
Distribution, "generating function"	one factor per recipient	"determine the generating function"
Palindromes of an integer	the left half decides	"palindromes of n"

2 Mistakes, "spot the error" and a checklist

Mistakes that cost marks

1. **Wrong bad condition.** "$x_i < 7$" is broken by $x_i \geq 7$; "$x_i \leq 5$" by $x_i \geq 6$.
2. **Forgetting to shift the bounds.** After $y = x - a$, the bound $x \leq b$ becomes $y \leq b - a$.
3. $S_k = 0$ **without a reason.** Write the one line: "three such variables already exceed the total".
4. **Assuming symmetry when the bounds differ.** Then each $N(c_i)$ must be computed separately.
5. **Mixing up** E_m **and** L_m **coefficients.** Check $L_m - E_m = L_{m+1}$.

6. $\binom{n}{m}(n-m)!$ **for "exactly m fixed points".** The others must avoid their own places.
7. **Double shifting in generating functions.** Either change the target or change the factor, not both.
8. **Palindrome words instead of palindromes of integers.**

Spot the error 1

"*Solution.*" Count $x_1 + x_2 + x_3 = 12$ with $0 \le x_i < 6$. The bad condition is $x_i > 6$. If $x_1 \ge 7$, the rest sum to 5: $\binom{7}{2} = 21$ solutions. Two variables ≥ 7 already exceed 12. So the answer is $\binom{14}{2} - 3 \cdot 21 = 28$.

Spot the error 2

"*Solution.*" An 8-question test is worth 100 points; each question is worth between 5 and 20 points, a multiple of 5. Give each question 5 points: 60 remain, and each $x_i \le 15$. Rescaling by 5: $y_1 + \cdots + y_8 = 12$ with $y_i \le 15$. Since $12 < 16$ the bound never matters, so the answer is $\binom{19}{7} = 50\,388$.

Spot the error 3

"*Solution.*" Over 60 days Alex had lunch with each of his 4 friends 20 times. Since $4 \cdot 20 = 80 > 60$, Alex never had lunch alone.

Spot the error 4

"*Solution.*" In BOOKKEEPER, glue the two O's into a block. The two O's inside the block can be ordered in 2! ways, so the number of arrangements with the O's together is $\dfrac{2! \cdot 9!}{2!\,2!\,3!}$.

Spot the error 5

"*Solution.*" Seven letters are put into their seven envelopes at random. Exactly 2 land in their own envelopes: choose the 2 in $\binom{7}{2}$ ways; the other 5 can be placed in 5! ways. Answer: $\binom{7}{2} \cdot 5! = 2520$.

Spot the error 6

"*Solution.*" 20 candies, 4 children, the youngest gets at least 3. Give the youngest 3 candies first, leaving 17. The youngest now gets at least 3 more, so the generating function is $(x^3 + x^4 + \cdots)(1 + x + \cdots)^3$ and the answer is the coefficient of x^{17}.

Spot the error 7

"*Solution.*" The palindromes of 7: write 7 as seven 1's; each of the 6 gaps gets a "+" or not, so there are $2^6 = 64$ of them.

Checklist

- ☐ I named the **universe** S and computed N.
- ☐ Every condition c_i is written down, and it is the **bad** event (bound $+1$).
- ☐ I shifted lower bounds **and** the matching upper bounds; I rescaled when all values are

multiples of d.
- ☐ Each S_k is justified; symmetry is used only when the bounds are equal.
- ☐ Every $S_k = 0$ has a one-line reason.
- ☐ For "exactly" / "at least" I used the right coefficients and checked $L_m = E_m + L_{m+1}$.
- ☐ Glued blocks of identical letters have **one** internal order.
- ☐ For fixed points, the non-fixed objects form a **derangement**.
- ☐ A generating function answer gives $f(x)$ and says which coefficient is needed.

3 Exercises

Write each solution in full. For inclusion and exclusion use the five steps: universe, conditions, the numbers S_k, combine, answer and check.

E1. How many integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ have with $0 \le x_i \le 10$ for all i?

E2. How many integer solutions does $x_1 + x_2 + x_3 = 15$ have with $2 \le x_1 \le 6$, $0 \le x_2 \le 8$, $1 \le x_3 \le 7$? Check your answer with a generating function.

E3. A 6-question quiz is worth 60 points; each question is worth between 5 and 15 points and a multiple of 5. In how many ways can the points be assigned?

E4. In a class of 40 students, 25 study French, 20 Spanish, 10 German; 10 study French and Spanish, 5 French and German, 4 Spanish and German, and 2 study all three. How many study none of the three? Exactly one?

E5. How many arrangements of AABBCCD have no two identical letters adjacent? Exactly one pair of adjacent identical letters? Check that your $E_0, \dots, E_3$ add up to the total.

E6. Prove that $\sum_{m=0}^{n} \binom{n}{m} d_{n-m} = n!$ by sorting the permutations of $1, \dots, n$ by their number of fixed points, and check it for $n = 4$.

E7. Prove that $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n \ge 2$ by looking at which envelope letter 1 goes into.

E8. Find the generating function, and then the number, of ways to distribute 24 pennies among 4 children if each child gets an even number of pennies and the youngest gets at least 6.

E9. List the palindromes of 7 and of 8, grouped by the sum of their left half.

E10. Show that the number of compositions of n is 2^{n-1} in two ways: the gap argument, and induction on n using the first summand.

A Answers to the "spot the error" puzzles

Spot the error 1 (the bound). "$x_i < 6$" allows $x_i = 0, \dots, 5$; it is broken already by $x_i = 6$. The bad condition is $x_i \ge 6$, not $x_i > 6$. Correct: $N(c_i) = \binom{8}{2} = 28$, and two variables ≥ 6 use up all 12 units (1 solution each), so $91 - 3 \cdot 28 + 3 \cdot 1 = 10$.

Spot the error 2 (rescaling the bound). After shifting by 5 the bound $x_i \le 20$ becomes $x_i - 5 \le 15$, and after dividing by 5 it is $y_i \le 3$, not $y_i \le 15$. The total was rescaled but the bound was not. Correct: PIE with $c_i : y_i \ge 4$, $\sum_{k=0}^{3} (-1)^k \binom{8}{k} \binom{19-4k}{7} = 50\,388 - 51\,480 + 9240 - 56 = 8092$.

Spot the error 3 (a sum is not a union). 80 counts a lunch with two friends twice and a lunch with three friends three times. A large total says nothing about whether some lunch had no friend: in principle all 80 meetings could have happened on a few days. Inclusion and exclusion, with the numbers of lunches with each pair, triple and all four friends, is needed.

Spot the error 4 (identical letters in a block). The two O's are identical, so the block OO has only one

order. After gluing there are 9 objects with K twice and E three times: $\frac{9!}{2!\,3!} = 30\,240$. (The "solution" multiplies by 2! and also divides by an extra 2! for the O's, which are no longer two separate letters; the two errors cancel, but the reasoning is wrong.)

Spot the error 5 (fixed points among the rest). Placing the other five letters in 5! ways allows some of them to land in their own envelopes, so arrangements with more than 2 fixed points are counted, several times. The five must form a derangement: $\binom{7}{2} d_5 = 21 \cdot 44 = 924$.

Spot the error 6 (double shift). The 3 candies are given twice: once by changing the target to 17, and again by the factor $x^3 + \cdots$. This counts distributions where the youngest gets at least 6. Use either target 20 with $x^3 + x^4 + \cdots$, or target 17 with $1 + x + \cdots$ for the youngest.

Spot the error 7 (palindromes are not all compositions). The gap argument counts *all* compositions of 7, such as $3 + 4$, which is not a palindrome. Only the compositions that read the same both ways count; by left halves (as for 6 in Section 1.7) there are $1 + 1 + 2 + 4 = 8$ of them.