Tutorial 6: Generating Functions, Partitions, Recurrences and Combinatorial Proofs

Weighted equations, partitions, exponential generating functions, solving recurrences, and counting in two ways

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Write the generating function of an equation with **coefficients**, such as $2u + 3v + 4w = n$, and handle **lower bounds** by pulling out a power of the variable.
2. Write the generating function for **partitions** of n with restrictions on how often a part may appear and on how large a part may be.
3. **Read off a sequence** from an exponential generating function, including ones that mix e^{ax} and polynomials.
4. Build the **exponential generating function** for **arrangements** of letters taken from a word with repeated letters, and explain why the $k!$ in the denominators is there.
5. **Solve a recurrence** with generating functions: multiply, sum, solve for $f(x)$, split into partial fractions, read the coefficient.
6. Write a **combinatorial proof** (count one set in two ways) and an **algebraic proof** of a binomial identity.

Sections 1.1 to 1.6 review the tools with the intuition, a simple example, the formal statement, a comprehensive example and a recap for each. Section 2 collects the classic mistakes, "spot the error" puzzles and a checklist, and Section 3 has practice exercises; the puzzles are answered in the appendix, so the notes work for **self-study**. Tutorials 1 to 5 (counting rules, $\binom{n}{r}$, stars and bars, induction, inclusion and exclusion, generating functions for distributions) are assumed.

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1 The toolbox

This tutorial is the generating function chapter. Sections 1.1 and 1.2 build **ordinary** generating functions for equations with coefficients and for partitions. Sections 1.3 and 1.4 introduce **exponential** generating functions, the right tool when *order* matters. Section 1.5 uses generating functions as a machine for **solving recurrences**, and Section 1.6 returns to binomial coefficients with **combinatorial proofs**. Each tool comes with the idea, a small example, the precise statement, a bigger example, and a recap.

1.1 Equations with coefficients: one factor per variable

The intuition

In Tutorial 5 every variable contributed 1 unit per step: the factor was $1 + t + t^2 + \cdots$. In $2u + 3v + 4w = n$ the variable u contributes 2 **units per step**: $2u$ can be $0, 2, 4, 6, \ldots$ So its factor is $1 + t^2 + t^4 + \cdots$, which is the geometric series in t^2:

$$1 + t^2 + t^4 + \cdots = \frac{1}{1 - t^2}.$$

Same rule as before: **one factor per variable, exponents = what that variable can contribute to the total**. A lower bound $u \geq 3$ means $2u \geq 6$, so the factor starts at t^6: $t^6 + t^8 + \cdots = \frac{t^6}{1-t^2}$.

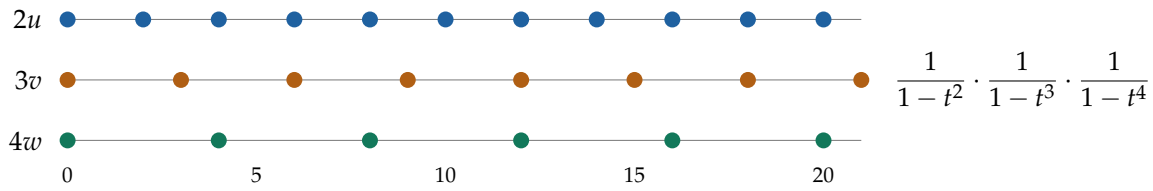


Figure 1. The possible contributions of each term of $2u + 3v + 4w$ to the total. Each row is one factor: its dots are the exponents that appear in it.

Simple example: making change

In how many ways can you make 15 cents with pennies, nickels and dimes? Let p, q, d be the numbers of coins: $p + 5q + 10d = 15$. The generating function is $\frac{1}{(1-t)(1-t^5)(1-t^{10})}$, and the answer is its coefficient of t^{15}. *By listing:* $d = 1$: $q \in \{0, 1\}$ (2 ways); $d = 0$: $q \in \{0, 1, 2, 3\}$ (4 ways). Total 6. ✓

Generating functions with coefficients

Let $a_1, \ldots, a_k$ be positive integers and $m_1, \ldots, m_k \geq 0$. The number of integer solutions of

$$a_1 y_1 + a_2 y_2 + \cdots + a_k y_k = n, \qquad y_i \geq m_i,$$

is the coefficient of t^n in

$$f(t) = \prod_{i=1}^{k} \left(t^{a_i m_i} + t^{a_i (m_i + 1)} + t^{a_i (m_i + 2)} + \cdots \right),$$

Each factor is a geometric series and can be summed in closed form. Use a letter for the generating function variable that is not one of the unknowns.

Comprehensive example: $3a + 4b + c = n$ with $a \geq 2, b \geq 1$

Solution. $3a \in \{6, 9, 12, \dots\}$ gives $\frac{t^6}{1-t^3}$; $4b \in \{4, 8, \dots\}$ gives $\frac{t^4}{1-t^4}$; $c \in \{0, 1, \dots\}$ gives $\frac{1}{1-t}$. So

$$f(t) = \frac{t^{10}}{(1-t)(1-t^3)(1-t^4)}.$$

Check. The smallest total is $3 \cdot 2 + 4 \cdot 1 + 0 = 10$, the exponent in front. For $n = 11$: $(a, b, c) = (2, 1, 1)$ only, and $[x^{11}]f = [x^1]\frac{1}{(1-t)(1-t^3)(1-t^4)} = 1$. ✓

Two slips with weighted variables

- **Lower bound on the variable, not on the term.** $v \geq 2$ in $3v$ means the term is at least 6: factor $\frac{t^6}{1-t^3}$, not $\frac{t^2}{1-t^3}$.
- **The same letter twice.** The generating function variable must be a fresh letter; if it coincides with one of the unknowns, a power such as "t^{12}" becomes ambiguous.

Weighted variables in one breath

- Term $a\,y$ with $y \geq m$: factor $\dfrac{t^{am}}{1-t^a}$.
- Multiply the factors; the answer for total n is $[x^n]$ of the product.

1.2 Partitions of an integer

The intuition

A **partition** of n writes n as a sum of positive integers where **order does not matter**: $3 + 1 + 1$ and $1 + 3 + 1$ are the same partition. (Compare compositions in Tutorial 5, where order matters.) Since order does not matter, a partition is described by **how many times each part is used**: y_1 ones, y_2 twos, y_3 threes, …, with

$$1 \cdot y_1 + 2 \cdot y_2 + 3 \cdot y_3 + \cdots = n.$$

That is an equation with coefficients, so Section 1.1 applies: the part i contributes the factor $1 + x^i + x^{2i} + \cdots$. Restrictions are restrictions on the y_i: for example "only odd parts" removes the factors for even i, and "each part at most once" keeps only $1 + x^i$.



$y_1 = 2$, $y_3 = 1$: the factor for 1 contributes x^2, the factor for 3 contributes x^3

Figure 2. Some partitions of 5 *drawn as dot diagrams (rows in decreasing order). Order does not matter, so a partition is just the list of how many times each part is used.*

Simple example: the partitions of 5

$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$: there are 7. The generating function $\prod_{i\geq 1} \frac{1}{1-x^i}$ starts $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \cdots$, and its coefficient of x^5 is 7. ✓

Generating functions for partitions

Let $p(n)$ be the number of partitions of n (with $p(0) = 1$). Then

$$\sum_{n\geq 0} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

Restrictions. Restrictions on the number y_i of copies of part i change the i-th factor:

Restriction	Factor for part i
no restriction	$1 + x^i + x^{2i} + \cdots = \frac{1}{1-x^i}$
part i used at most k times	$1 + x^i + \cdots + x^{ki}$
distinct parts (at most once)	$1 + x^i$
part i not allowed	1 (drop the factor)

Comprehensive example: distinct parts versus odd parts

Show that the number of partitions of n into **distinct** parts equals the number into **odd** parts.
Solution. Distinct parts: $\prod_{i\geq 1}(1 + x^i)$. Odd parts: $\prod_{j\geq 1} \frac{1}{1-x^{2j-1}}$. Now

$$\prod_{i\geq 1}(1+x^i) = \prod_{i\geq 1} \frac{1-x^{2i}}{1-x^i} = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},$$

since every even factor in the denominator cancels with the numerator. Equal generating functions mean equal coefficients. *Check* ($n = 6$): distinct: $6, 5+1, 4+2, 3+2+1$; odd: $5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1$. Four each. ✓

Restrictions on how often a part repeats are not restrictions on the number of parts

"Distinct parts" limits how often *each* value repeats: $4 + 2 + 1$ is fine (three parts, all different), but $2 + 2 + 1$ is not. It does *not* say "at most one summand". Always ask: is the condition about each part size, or about the number of summands? The two give different generating functions.

Partitions in one breath

- One factor per part size i; its exponents are the multiples of i allowed.
- A restriction on part i changes only the factor for part i; a forbidden part loses its factor.

1.3 Exponential generating functions: reading off a sequence

The intuition

An ordinary generating function stores a_n as the coefficient of x^n. An **exponential** generating function stores a_n as the coefficient of $\frac{x^n}{n!}$. It is the same idea with a different "filing system", and it is chosen because the most important function in it is very simple:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{is the EGF of} \quad 1, 1, 1, 1, \ldots$$

To read a_n from an EGF $f(x)$, find the coefficient of x^n and **multiply by** $n!$: $a_n = n! \cdot [x^n]f(x)$.

EGF $f(x)$	coefficient of x^n	sequence $a_n = n! \cdot [x^n]f$
e^x	$\frac{1}{n!}$	$1, 1, 1, 1, \ldots$
e^{ax}	$\frac{a^n}{n!}$	$1, a, a^2, a^3, \ldots$
$c\,x^k$ (one term)	c at $n = k$, else 0	$c \cdot k!$ at position k, 0 elsewhere
$\frac{1}{(1-x)^2} = \sum_n (n+1)x^n$	$n+1$	$(n+1)\,n! = (n+1)!$: $1, 2, 6, 24, \ldots$

Table 1. A small dictionary. In the last row the ordinary coefficient $n+1$ is multiplied by $n!$.

Simple example: $f(x) = e^{2x} + x$

e^{2x} gives 2^n. The single term $x = 1 \cdot \frac{x^1}{1!}$ adds 1 at position 1. So the sequence is $1,\ 2+1 = 3,\ 4,\ 8,\ 16, \ldots$, that is $a_1 = 3$ and $a_n = 2^n$ for $n \neq 1$.

Exponential generating function

The **exponential generating function** (EGF) of $a_0, a_1, a_2, \ldots$ is

$$f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots$$

Linearity. If f is the EGF of (a_n) and g of (b_n), then $\alpha f + \beta g$ is the EGF of $(\alpha a_n + \beta b_n)$. So split f into pieces, read each piece, and add position by position. **Polynomials.** A term $c\,x^k$ equals $c\,k! \cdot \frac{x^k}{k!}$, so it adds $c\,k!$ to a_k only.

Comprehensive example: $f(x) = 2e^{-x} + 4x^3 + \frac{1}{(1-x)^2}$

n	0	1	2	3	4	general n
$2e^{-x}$	2	-2	2	-2	2	$2(-1)^n$
$4x^3$	0	0	0	$4 \cdot 3! = 24$	0	
$\frac{1}{(1-x)^2}$	1	2	6	24	120	$(n+1)!$
a_n	3	0	8	46	122	$2(-1)^n + (n+1)!$ for $n \neq 3$

Forgetting the $n!$

The coefficient of x^n is *not* a_n. For $e^{-x} + x^4$ the coefficient of x^4 is $\frac{1}{24} + 1 = \frac{25}{24}$, and $a_4 = 4! \cdot \frac{25}{24} = 25$, not $\frac{25}{24}$ and not 2. Always rewrite each term as (something) $\cdot \frac{x^n}{n!}$ before reading.

Reading an EGF in one breath

- $a_n = n! \cdot [x^n]f$. Split f into pieces, read each, add.
- $e^{ax} \mapsto a^n$; $cx^k \mapsto c\,k!$ at $n = k$.

1.4 Exponential generating functions for arrangements

The intuition

To count **selections** of letters (order does not matter), an ordinary generating function works: the letter A that may be used 0, 1 or 2 times contributes $1 + x + x^2$. To count **arrangements** (order matters), we need the number of words, and a word that uses A twice, B once and C once has $\frac{4!}{2!\,1!\,1!}$ arrangements. The $\frac{1}{2!}$ is about the A's alone, so it can be built into A's factor: $1 + x + \frac{x^2}{2!}$. When the factors are multiplied, the denominators multiply to $2!\,1!\,1!$, and reading the coefficient of $\frac{x^4}{4!}$ supplies the 4! on top. That is exactly the EGF.

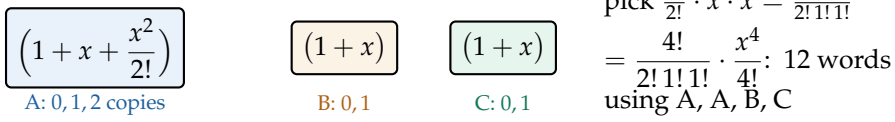


Figure 3. Why EGFs count arrangements. Each choice of one term per letter is a choice of how many copies of each letter to use, and it contributes the number of distinct arrangements of those copies.

Simple example: two letters from AAB

Words of length 2 using letters of AAB: AA, AB, BA. That is 3.
EGF: $\left(1 + x + \frac{x^2}{2!}\right)(1 + x) = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3$. Coefficient of x^2 is $\frac{3}{2}$, times 2!: 3. ✓
(The ordinary GF $(1 + x + x^2)(1 + x)$ gives 2 at x^2: the *selections* $\{A, A\}, \{A, B\}$.)

EGF for arrangements

Suppose there are r kinds of objects and objects of kind j may be used k_j times for k_j in an allowed set K_j. The number of arrangements (words) of length n is $n! \cdot [x^n]$ of

$$f(x) = \prod_{j=1}^{r} \Big(\sum_{k \in K_j} \frac{x^k}{k!} \Big).$$

Reason. A choice of exponents $k_1 + \cdots + k_r = n$ contributes $\frac{x^n}{k_1! \cdots k_r!} = \binom{n}{k_1,\ldots,k_r} \frac{x^n}{n!}$, and $\binom{n}{k_1,\ldots,k_r} = \frac{n!}{k_1! \cdots k_r!}$ is the number of arrangements with exactly those multiplicities.
Typical factors: at most m copies: $1 + x + \cdots + \frac{x^m}{m!}$; at least 1 copy: $x + \frac{x^2}{2!} + \cdots = e^x - 1$; any number: e^x; an even number: $\frac{e^x + e^{-x}}{2}$.

Comprehensive example: three letters from BANANA

BANANA has B once, A three times, N twice. So

$$f(x) = (1 + x)\Big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\Big)\Big(1 + x + \frac{x^2}{2!}\Big).$$

The terms giving x^3, with their multiplicities (B, A, N): $(1,2,0)$: $\frac{1}{2}$; $(1,0,2)$: $\frac{1}{2}$; $(1,1,1)$: 1; $(0,3,0)$: $\frac{1}{6}$; $(0,2,1)$: $\frac{1}{2}$; $(0,1,2)$: $\frac{1}{2}$. Sum $\frac{19}{6}$, so there are $3! \cdot \frac{19}{6} = 19$ words of length 3.
Check by complement. All $3^3 = 27$ words over $\{B, A, N\}$, minus those with ≥ 2 B's ($3 \cdot 2 + 1 = 7$), minus NNN (1): $27 - 8 = 19$. ✓

Selections versus arrangements

"In how many ways can we *arrange* n letters selected from …" asks for words: EGF. "In how many ways can we *select* …" asks for multisets: ordinary GF. Using $1 + x + x^2$ for a letter in an arrangement question is the most common mistake with this tool.

EGFs for arrangements in one breath

- Letter allowed $0, \ldots, m$ times: $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!}$. Multiply over the letter types.
- Number of words of length n: $n! \cdot [x^n] f$. Check: the full length gives $\frac{(\text{length})!}{\prod(\text{multiplicities})!}$.

1.5 Solving recurrences with generating functions

The intuition

A recurrence is infinitely many equations, one for each n. A generating function packs the whole sequence into one function $f(x) = \sum a_n x^n$. So: **multiply** the n-th equation by a power of x, **add** all of them, and you get *one* equation for $f(x)$. Solve it with ordinary algebra, then **unpack** $f(x)$ with partial fractions and the series you know. The initial values enter when you rewrite the shifted sums in terms of $f(x)$.

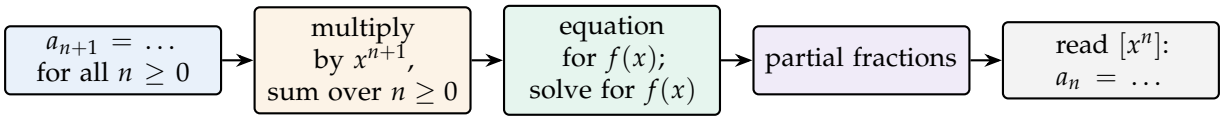


*Figure 4. The five steps of the method. Multiply by the **highest** power that appears (x^{n+1} for a first-order recurrence, x^{n+2} for a second-order one), so every sum starts cleanly.*

Simple example: $a_{n+1} = 2a_n$, $a_0 = 3$

Multiply by x^{n+1} and sum over $n \geq 0$: $\sum_{n\geq 0} a_{n+1}x^{n+1} = 2x\sum_{n\geq 0} a_n x^n$. The left sum is $f(x)$ without its first term: $f(x) - a_0 = 2xf(x)$. So $f(x)(1-2x) = 3$, $f(x) = \frac{3}{1-2x} = \sum 3\cdot 2^n x^n$, and $a_n = 3\cdot 2^n$. ✓

The shifted sums and the series you need

With $f(x) = \sum_{n\geq 0} a_n x^n$:

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = f(x) - a_0, \qquad \sum_{n\geq 0} a_{n+2}x^{n+2} = f(x) - a_0 - a_1 x, \qquad \sum_{n\geq 0} a_n x^{n+k} = x^k f(x).$$

Series to recognise.

function	coefficient of x^n	function	coefficient of x^n
$\frac{1}{1-ax}$	a^n	$\frac{x}{(1-x)^2}$	n
$\frac{1}{(1-x)^k}$	$\binom{n+k-1}{k-1}$		

The series $\frac{x}{(1-x)^2}$ comes from $\frac{1}{1-x} = \sum x^n$ by "differentiate, then multiply by x". **Partial fractions.** For distinct $a \neq b$: $\frac{p(x)}{(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx}$; find A by putting $x = \frac{1}{a}$ in $p(x) = A(1-bx) + B(1-ax)$, and B with $x = \frac{1}{b}$. A repeated factor $(1-x)^k$ needs the terms $\frac{A_1}{1-x} + \dots + \frac{A_k}{(1-x)^k}$.

Comprehensive example: $a_{n+2} = a_{n+1} + 2a_n$, $a_0 = 0$, $a_1 = 3$

Solution. Multiply by x^{n+2} and sum over $n \geq 0$:

$$\big(f(x) - a_0 - a_1 x\big) = x\big(f(x) - a_0\big) + 2x^2 f(x),$$

so $f(x)(1 - x - 2x^2) = 3x$ and $f(x) = \dfrac{3x}{(1-2x)(1+x)}$. Partial fractions: $3x = A(1+x) + B(1-2x)$. With $x = \frac{1}{2}$: $\frac{3}{2} = \frac{3}{2}A$, so $A = 1$. With $x = -1$: $-3 = 3B$, so $B = -1$. Hence $f(x) = \frac{1}{1-2x} - \frac{1}{1+x}$ and $a_n = 2^n - (-1)^n$.
Check. $a_0 = 0$, $a_1 = 3$, $a_2 = 3 = a_1 + 2a_0$, $a_3 = 9 = a_2 + 2a_1$. ✓

Two slips in the method

- **Dropping the initial values.** $\sum_{n\geq 0} a_{n+1}x^{n+1}$ is $f(x) - a_0$, not $f(x)$. Without $-a_0$ the answer cannot depend on a_0, which is clearly wrong.
- **Shifting the right-hand side wrongly.** A constant right-hand side 1 gives $\sum_{n\geq 0} x^{n+1} = \frac{x}{1-x}$,

not $\frac{1}{1-x}$: the extra x comes from multiplying by x^{n+1}, not x^n.

Recurrences by generating functions in one breath

- Multiply by $x^{n+(\text{order})}$, sum, replace shifted sums by $f(x)$ minus initial terms, solve for $f(x)$.
- Partial fractions, then read a_n from known series. Check a_0, a_1, a_2 against the recurrence.

1.6 Combinatorial proofs

The intuition

To prove LHS = RHS for two counting expressions, find **one set** and count it **in two different ways**: one way gives the LHS, the other gives the RHS. Both count the same set, so they are equal. No algebra is needed, and the proof often explains *why* the identity is true.
A sum on one side usually means: **split the set into cases** and add (rule of sum). A product on the other side usually means: build the object **in stages** (rule of product).

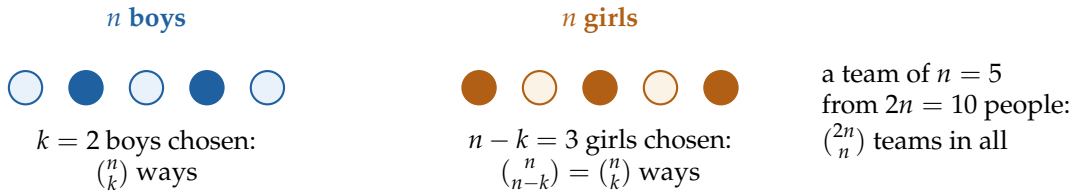


Figure 5. Counting the teams of n people from n boys and n girls. All at once: $\binom{2n}{n}$. Split by the number k of boys in the team: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$ for each k.

Simple example: $\binom{n}{k} = \binom{n}{n-k}$

Count the k-person teams from n people. Way 1: choose the team, $\binom{n}{k}$. Way 2: choose the $n-k$ people who are *left out*, $\binom{n}{n-k}$; the team is everyone else. Both count teams, so they are equal.

How to write a combinatorial proof

(i) **Define the set.** Describe precisely which objects are counted ("the n-element subsets of a set of n boys and n girls").
(ii) **Count it one way.** Justify each factor with the rule of product or sum; this gives the LHS.
(iii) **Count it another way.** Again justify each factor; this gives the RHS.
(iv) **Conclude.** "Since both expressions count the same set, they are equal."
An **algebraic proof** instead rewrites one side into the other using definitions such as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
Both are valid proofs; a question that asks for one of them wants that one.

Comprehensive example: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Solution. Count the teams of n people chosen from a group of n boys and n girls. *Way 1.* Choose n of the $2n$ people: $\binom{2n}{n}$. *Way 2.* Split by the number k of boys in the team, $0 \le k \le n$ (rule of sum). For a fixed k, choose the boys ($\binom{n}{k}$ ways) and the $n-k$ girls ($\binom{n}{n-k}$ ways), so $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$

teams (rule of product, then symmetry). Adding over k gives the LHS. Both count the same teams. ■
Check ($n = 3$): $1 + 9 + 9 + 1 = 20 = \binom{6}{3}$. ✓

The two counts must count the same set

A common slip is to describe two procedures that count *different* things, for example "choose k boys and k girls" in the proof above: that builds teams of $2k$ people, not teams of n. Name the set once, at the start, and check that each procedure produces every element of it exactly once.

Combinatorial proofs in one breath

- One set, two counts. Name the set first.
- A sum $\sum_k$ on one side: split the set by a parameter k. A product: build in stages.

The toolbox at a glance

Situation	Count / method	Signal words
Equation with coefficients	factor $\frac{t^{am}}{1-t^a}$ per term ay, $y \geq m$	"$2u + 3v + 4w = n$"
Partitions with restrictions	factor per part size i	"partitions of n into summands"
Reading an EGF	$a_n = n![x^n]f$	"sequence generated by the exponential…"
Arrangements of letters	$\prod\left(1 + x + \cdots + \frac{x^m}{m!}\right)$	"arrange n letters selected from"
Recurrence	multiply, sum, solve, split, read	"by the method of generating functions"
Identity with binomials	one set, two counts; or factorials	"combinatorial proof", "algebraic proof"

2 Mistakes, "spot the error" and a checklist

Mistakes that cost marks

1. **Lower bound on the variable treated as a bound on the term.** $v \geq 2$ in $3v$ gives t^6, not t^2.
2. **Using one letter for two things**: the generating function variable must not be an unknown.
3. **A per-part restriction read as a restriction on the number of parts**, or the other way round.
4. **Reading $[x^n]$ as a_n in an EGF.** Multiply by $n!$; the polynomial terms change by $k!$.
5. **Ordinary factors in an arrangement problem.** The factors need $\frac{x^k}{k!}$.
6. **Miscounting the letters** of a word before building its factors.
7. **Dropping initial values** when replacing shifted sums by $f(x)$.
8. **Two procedures for two different sets** in a combinatorial proof.

Spot the error 1

"*Solution.*" For $2u + 3v + 4w = n$ with $u \geq 0$, $v \geq 2$, $w \geq 1$, the generating function is $\dfrac{1}{1-t^2}$.

$$\frac{t^2}{1-t^3}\cdot\frac{t}{1-t^4}=\frac{t^3}{(1-t^2)(1-t^3)(1-t^4)}.$$

Spot the error 2

"*Solution.*" For partitions of n into *distinct odd* parts, each odd part $2j-1$ may be used, so the generating function is $\prod_{j\geq 1}\frac{1}{1-x^{2j-1}}$.

Spot the error 3

"*Solution.*" $e^{2x}+x^4$: e^{2x} generates $1,2,4,8,16,\ldots$ and x^4 adds 1 at position 4, so the sequence is $1,2,4,8,17,32,\ldots$

Spot the error 4

"*Solution.*" $\frac{1}{(1-x)^2}=\sum_i(i+1)x^i=\sum_i(i+1)x^i\cdot\frac{i!}{i!}=(i+1)!\sum_i\frac{x^i}{i!}$, so the EGF generates $(n+1)!$.

Spot the error 5

"*Solution.*" For arrangements of letters from AABC: A can be used 0, 1 or 2 times, B and C 0 or 1 times, so the exponential generating function is $(1+x+x^2)(1+x)^2$.

Spot the error 6

"*Solution.*" Solve $a_{n+1}+2a_n=1$, $a_0=2$. Summing $a_{n+1}x^{n+1}+2a_nx^{n+1}=x^{n+1}$ over $n\geq 0$ gives $f(x)+2xf(x)=\frac{x}{1-x}$, so $f(x)=\frac{x}{(1-x)(1+2x)}$ and $a_n=\frac{1-(-2)^n}{3}$.

Spot the error 7

"*Proof of* $\sum_k\binom{n}{k}^2=\binom{2n}{n}$." On the left, choose k boys and k girls from n boys and n girls, in $\binom{n}{k}\binom{n}{k}$ ways, and add over k. On the right, choose n of the $2n$ people.

Checklist

- ☐ The generating function variable is **not** one of the unknowns.
- ☐ For each weighted variable, the factor starts at (coefficient) × (lower bound).
- ☐ For partitions, each factor matches the restriction on **that part size**; the product stops at the largest allowed part.
- ☐ When reading an EGF, I multiplied by $n!$ and handled each polynomial term separately.
- ☐ For arrangements I used $\frac{x^k}{k!}$, and the full-length coefficient gives $\frac{(\text{length})!}{\prod m_j!}$.
- ☐ In each recurrence, I multiplied by the right power of x and **subtracted the initial terms**.
- ☐ My partial-fraction constants were checked by plugging in a value of x, and I verified a_0, a_1, a_2.
- ☐ Each combinatorial proof names **one set** and counts it twice, with every factor justified.

3 Exercises

Write each solution in full: one line per factor (and why), or every step of the recurrence method, or a named set counted twice.

E1. Find the generating function for the number of integer solutions of $2a + 3b + 4c = n$ with $a \geq 1$, $b \geq 0$, $0 \leq c \leq 2$.

E2. In how many ways can you make 25 cents with pennies, nickels, dimes and quarters? Write the generating function, then count by cases.

E3. Write the generating functions for partitions of n into (a) distinct parts, (b) odd parts, (c) parts of size at most 4, (d) parts each used at most twice. Find the coefficient of x^8 in (a) and (b).

E4. Determine the sequence generated by each EGF: (a) e^{-x}; (b) xe^x; (c) $\frac{e^x+e^{-x}}{2}$; (d) $5e^x - 2x^4$.

E5. Find the EGF for the number of arrangements of n letters selected from BANANA, and use it to count the arrangements of length 4 and of length 6.

E6. Use an EGF to show that the number of strings of length n over $\{0, 1, 2\}$ with an *even* number of 0s is $\frac{3^n+1}{2}$.

E7. Solve $a_{n+1} - 2a_n = 1$, $n \geq 0$, $a_0 = 0$ by generating functions.

E8. Solve $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $n \geq 0$, $a_0 = 1$, $a_1 = 4$ by generating functions.

E9. Give a combinatorial proof of the hockey-stick identity $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ (hint: sort the $(k+1)$-subsets of $\{1, \ldots, n+1\}$ by their largest element).

E10. Give a combinatorial proof of Pascal's identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $1 \leq k \leq n-1$ (hint: does the team contain person 1?), and then an algebraic proof.

A Answers to the "spot the error" puzzles

Spot the error 1 (bound on the variable). $v \geq 2$ means the *term* $3v \geq 6$, so its factor is $\frac{t^6}{1-t^3}$; in the same way $4w \geq 4$. Correct: $\dfrac{t^{10}}{(1-t^2)(1-t^3)(1-t^4)}$.

Spot the error 2 ("distinct" was ignored). $\prod_j \frac{1}{1-x^{2j-1}}$ allows each odd part any number of times: it counts partitions into odd parts, such as $3+3$. Distinct odd parts means each odd part at most once: $\prod_{j\geq 1}(1+x^{2j-1})$. For $n = 6$ the first gives 4 partitions, the correct one only $5+1$.

Spot the error 3 (the 4!). $x^4 = 24 \cdot \frac{x^4}{4!}$, so the term adds 24 (not 1) to a_4. Correct: $1, 2, 4, 8, 40, 32, 64, \ldots$

Spot the error 4 (pulling $(i+1)!$ out of the sum). The summation index i cannot be moved outside the sum: "$(i+1)!\sum_i$" has no meaning. The correct step keeps it inside: $\sum_i (i+1)x^i = \sum_i (i+1)!\,\frac{x^i}{i!}$, so the coefficient of $\frac{x^i}{i!}$ is $(i+1)!$. The answer is right, the reasoning is not.

Spot the error 5 (selections, not arrangements). $(1+x+x^2)(1+x)^2$ is an ordinary generating function: its coefficient of x^4 is 1, the single *selection* of all four letters. AABC has $\frac{4!}{2!} = 12$ *arrangements*. Correct: $\left(1+x+\frac{x^2}{2!}\right)(1+x)^2$.

Spot the error 6 (the initial value). $\sum_{n\geq 0} a_{n+1}x^{n+1} = f(x) - a_0$, not $f(x)$. The formula gives $a_0 = 0 \neq 2$. Correct: $(1+2x)f(x) = 2 + \frac{x}{1-x} = \frac{2-x}{1-x}$, and partial fractions give $a_n = \frac{1}{3} + \frac{5}{3}(-2)^n$ (check: $a_1 = -3 = 1 - 2\cdot 2$, $a_2 = 7 = 1 - 2\cdot(-3)$).

Spot the error 7 (two different sets). Choosing k boys and k girls builds teams of $2k$ people, which are not the teams of n people counted on the right. Choose k boys and $n-k$ girls instead: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$.