Tutorial 9: Isomorphism, Complements, Hypercubes and Longest Paths

invariants, self-complementary graphs, the hypercube Q_n and Hamming distance, counting paths,
building graphs with given degrees, the longest-path trick, and paths in bipartite graphs

COMP 339 tutorial notes (by Parsa Kamalipour)

What you will be able to do after this tutorial

1. Decide whether two graphs are **isomorphic**: exhibit a bijection to say yes, find an **invariant** that differs to say no.
2. Work with the **complement** $\overline{G}$, and verify that a graph is **self-complementary**.
3. Describe the **hypercube** Q_n three ways (bit strings, recursion, degrees), count its edges, and compute distances in it with the **Hamming distance**.
4. Build a **Gray code**: a path through *all* vertices of Q_n.
5. **Count** paths of a given length by choosing the middle vertex first.
6. **Construct** a graph with prescribed degrees and *verify* the construction.
7. Use the **longest path trick** to force structure from a minimum-degree condition.
8. Bound the length of paths in a **bipartite** graph by the alternation argument.

Section 1 is the toolbox: each topic has the intuition, a simple example, the formal statement, a comprehensive example and a recap. Section 2 collects common mistakes and "spot the error" puzzles, and Section 3 has practice problems. Answers to everything are in the appendix, so the notes work for **self-study**. Tutorials 7 and 8 (degrees, Handshaking, bipartite graphs, Hamilton paths) are assumed.

Conventions used throughout

A **graph** is finite, undirected and **simple** (no loops, no multiple edges) unless stated otherwise; $v = |V|$, $e = |E|$. The **length** of a path is its number of **edges**, so a path of length k has $k + 1$ distinct vertices. $N(x)$ is the set of neighbours of x, and $\delta(G)$ is the minimum degree.

Contents

1 The toolbox **2**

1.1 Isomorphism and invariants . . . 2
1.2 Complements and self-complementary graphs . . . 3
1.3 The hypercube Q_n and Hamming distance . . . 5
1.4 Gray codes: a path through every vertex of Q_n . . . 6
1.5 Counting paths: choose the middle first . . . 7
1.6 Building a graph with prescribed degrees . . . 8
1.7 The longest path trick . . . 9
1.8 Paths in bipartite graphs . . . 10

2 Mistakes and "spot the error" **11**

3 Practice problems 11

A Answers to the "spot the error" puzzles 12

B Answers to the practice problems 12

1 The toolbox

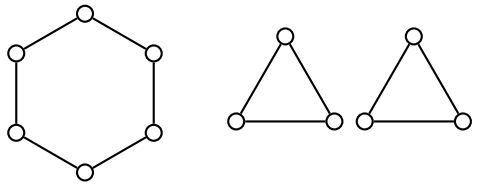
1.1 Isomorphism and invariants

The intuition

Two graphs are **isomorphic** when they are the same graph with the vertices renamed: the drawing may look completely different, but who-is-joined-to-whom is identical. So there are two kinds of answer:

- **"Yes, isomorphic"**: give the renaming f and check it sends edges to edges and non-edges to non-edges.
- **"No, not isomorphic"**: find a property that renaming cannot change (an **invariant**) on which the two graphs differ.

Trying all $n!$ renamings is never the answer.



C_6: connected $2K_3$: two components

Same $v = 6$, same $e = 6$, same degrees $2, 2, 2, 2, 2, 2$, and still **not** isomorphic: C_6 is connected, $2K_3$ is not; $2K_3$ has triangles, C_6 has none. One differing invariant is a complete proof.

Figure 1. Matching counts do not prove isomorphism. The degree sequence is an invariant, but it is not a complete one.

Simple example: a renaming that works

G is the path $a - b - c$ and H is the path $x - z - y$. The map $f(a) = x, f(b) = z, f(c) = y$ sends the edges ab, bc to xz, zy, the edges of H, and the non-edge ac to the non-edge xy. So $G \cong H$. Note that $f(b) = z$ is forced: the middle vertex must go to the middle vertex, since degree is preserved.

Isomorphism and the standard invariants

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic**, $G_1 \cong G_2$, if there is a **bijection** $f : V_1 \to V_2$ such that for all $a, b \in V_1$,

$$\{a, b\} \in E_1 \iff \{f(a), f(b)\} \in E_2.$$

If $G_1 \cong G_2$ then they agree on every **invariant**, including: the number of vertices and of edges; the degree sequence; connectedness and the number of components; the number of triangles (or of cycles of any given length); bipartiteness; planarity; the length of a longest path. Moreover f preserves distances: $d_{G_2}(f(a), f(b)) = d_{G_1}(a, b)$.

Comprehensive example: same degrees, different graphs, found by a finer invariant

Let G be the triangular prism (two triangles $a_1a_2a_3$, $b_1b_2b_3$ plus the edges a_ib_i) and $H = K_{3,3}$. Both have 6 vertices, 9 edges, and every degree is 3. But G contains the triangle $a_1a_2a_3$, while $K_{3,3}$ is bipartite and so contains **no** odd cycle at all (Tutorial 8). Number of triangles is an invariant, so $G \not\cong H$. (Tutorial 8 found another difference: one is planar and the other is not.)

Equal counts are not a proof

"Both have 6 vertices, 9 edges and the same degrees, so they are isomorphic" is the most common wrong argument in this chapter. Equal invariants only mean that you have not yet found a difference. To prove **yes**, write the bijection.

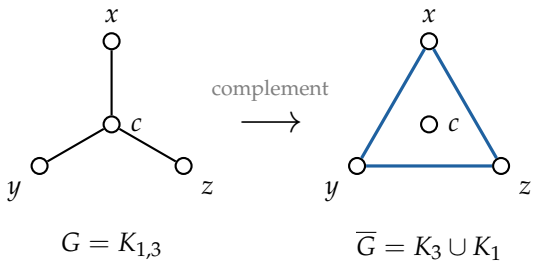
Isomorphism in one breath

- Yes: give f, check edges $\leftrightarrow$ edges (and count: equal e plus "edges to edges" is enough, since f is a bijection on pairs).
- No: one invariant that differs (components, degrees, triangles, bipartite, longest path, ...).

1.2 Complements and self-complementary graphs

The intuition

Start from K_n and paint each of its edges either black (in G) or white (not in G). The white edges form the **complement** $\overline{G}$. So G and $\overline{G}$ share the $\binom{n}{2}$ pairs between them, and a vertex of degree d in G has degree $n-1-d$ in $\overline{G}$.
A graph is **self-complementary** when its white picture is the same graph as its black picture, only with the vertices renamed. To prove it you give a renaming that turns every black pair white and every white pair black.



$G = K_{1,3}$ $\qquad$ $\overline{G} = K_3 \cup K_1$

$e(G) + e(\overline{G}) = 3 + 3 = 6 = \binom{4}{2}$ ✓
$\deg_{\overline{G}}(c) = 3 - 3 = 0$, $\deg_{\overline{G}}(x) = 3 - 1 = 2$ ✓
Here G is connected and $\overline{G}$ is not:
G and $\overline{G}$ need not look alike at all.

Figure 2. The complement of the star $K_{1,3}$. The centre, joined to everything in G, is isolated in $\overline{G}$.

Simple example: the complement of C_6

In C_6 with vertices $0, \ldots, 5$, vertex i is joined to $i \pm 1$. In $\overline{C_6}$ it is joined to the other three: $i+2$, $i+3$, $i+4$ (mod 6). The edges $\{i, i+2\}$ form two triangles $0, 2, 4$ and $1, 3, 5$, and the edges $\{i, i+3\}$ join them by a matching. So $\overline{C_6}$ is the triangular prism. Check: $6 + 9 = 15 = \binom{6}{2}$.

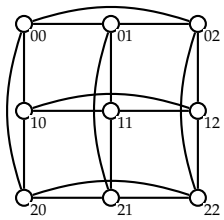
Complement, self-complementary

For $G = (V, E)$ with $|V| = n$, the **complement** is $\overline{G} = (V, \overline{E})$ where $\{x, y\} \in \overline{E}$ if and only if $x \neq y$ and $\{x, y\} \notin E$. Then

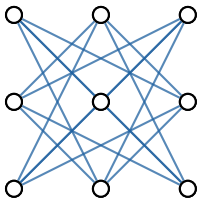
$$|E| + |\overline{E}| = \binom{n}{2}, \qquad \deg_{\overline{G}}(x) = n - 1 - \deg_G(x), \qquad \overline{\overline{G}} = G.$$

Fact. $G \cong H$ if and only if $\overline{G} \cong \overline{H}$ (the same bijection works, because it sends non-edges to non-edges).
G is **self-complementary** if $G \cong \overline{G}$, that is, there is a bijection $f : V \to V$ with

$$\{a, b\} \in E \iff \{f(a), f(b)\} \notin E \qquad \text{for all } a \neq b.$$



R: $(i, j) \sim (i', j')$ iff same row *or* same column



$\overline{R}$: different row *and* different column

9 vertices, deg $= 4$ in both, $18 + 18 = 36 = \binom{9}{2}$.

The renaming $f(i, j) = (i + j,\ i - j) \bmod 3$ turns every edge of R into an edge of $\overline{R}$ and back.

Figure 3. The 3×3 rook graph R ($K_3 \,\square\, K_3$: rooks attack along rows and columns) is self-complementary.

Comprehensive example: the 3×3 rook graph is self-complementary

Vertices are pairs (i, j) with $i, j \in \{0, 1, 2\}$; two are adjacent in R when they share a row or a column (Figure 3). Define $f(i, j) = (i + j,\ i - j)$ with arithmetic mod 3.
f is a bijection: from $a = i + j$ and $b = i - j$ we recover $i = 2(a + b)$ and $j = 2(a - b)$ mod 3 (because $2 \cdot 2 = 4 \equiv 1$), so f has an inverse.
Edges go to non-edges. Write $\Delta i, \Delta j$ for the differences between two distinct vertices; the images differ by $(\Delta i + \Delta j,\ \Delta i - \Delta j)$.

- Same row ($\Delta i = 0$, $\Delta j \neq 0$): the image difference is $(\Delta j, -\Delta j)$, both coordinates non-zero, so the images are in different rows and columns: an edge of $\overline{R}$. Same column: similarly $(\Delta i, \Delta i)$.
- Both differences non-zero (a non-edge of R): in $\mathbb{Z}_3$ either $\Delta j = \Delta i$, making the second coordinate 0, or $\Delta j = -\Delta i$, making the first 0. Either way the images share a row or a column: a non-edge of $\overline{R}$.

So $\{a, b\} \in E(R) \iff \{f(a), f(b)\} \notin E(R)$, and R is self-complementary. ■

Two traps

(i) "G and $\overline{G}$ have the same number of edges, so G is self-complementary" is wrong: equal edge counts are necessary, not sufficient. You must exhibit f.
(ii) The map f goes from G to $\overline{G}$: check that it sends edges of G to edges of $\overline{G}$, not to edges of G.

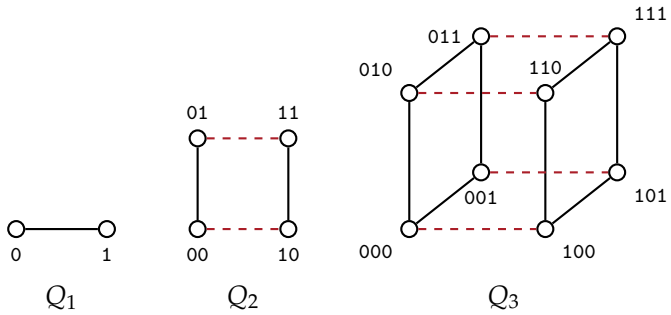
Complements in one breath

- $e(G) + e(\overline{G}) = \binom{n}{2}$; $\deg_{\overline{G}} = n - 1 - \deg_G$; $G \cong H \iff \overline{G} \cong \overline{H}$.
- Self-complementary: a bijection that swaps edges and non-edges. Prove it by checking both kinds of pair.

1.3 The hypercube Q_n and Hamming distance

The intuition

Write every vertex as a string of n bits, and join two strings when they differ in **exactly one** bit. That is the n-cube Q_n. A walk in Q_n is a sequence of single bit-flips, so travelling from x to y means flipping, one at a time, every bit where x and y disagree. The distance is therefore just the number of disagreeing positions, the **Hamming distance**.
The second view is recursive: Q_n is two copies of Q_{n-1} (strings starting with 0 and with 1) joined by the edges that flip the first bit. That view is what makes induction on n work.



Q_n has 2^n vertices.
Each vertex has n bits to flip, so Q_n is n-**regular**.
Dashed red: the edges that flip the **first** bit. Deleting them leaves two copies of Q_{n-1}.

Figure 4. Q_1, Q_2, Q_3. Each is two copies of the previous one (black) plus a perfect matching (red) between twins.

Simple example: counting the edges of Q_3 and Q_4

Q_3 has 8 vertices of degree 3, so by the Handshaking Theorem $2e = 8 \cdot 3$ and $e = 12$, the 12 edges of a cube. Q_4 has 16 vertices of degree 4: $2e = 64$, $e = 32$. Recursively: $e(Q_4) = 2e(Q_3) + 8 = 24 + 8 = 32$ (two copies plus one matching edge per vertex of Q_3). The two methods agree. ✓

Hypercube, Hamming distance

For $n \geq 1$, the **hypercube** Q_n has vertex set $\{0,1\}^n$, and $x \sim y$ exactly when x and y differ in exactly one coordinate. It has 2^n vertices, is n-regular, is bipartite (Tutorial 8: colour by the parity of the number of 1s), and is connected.

For a d-regular graph on v vertices the Handshaking Theorem gives $e = \dfrac{d\,v}{2}$.

The **Hamming distance** $h(x,y)$ is the number of coordinates in which x and y differ.
Theorem. In Q_n, $d(x,y) = h(x,y)$ for all vertices x, y.
Proof. ($\leq$) Flip the $h(x,y)$ disagreeing bits one at a time; each flip is an edge, so this is a walk of length $h(x,y)$ from x to y. ($\geq$) Each edge changes exactly one bit, so it lowers the number of disagreements with y by at most 1; starting from $h(x,y)$ disagreements, at least $h(x,y)$ edges are needed to reach 0. ■



$h(0110, 0000) = 2$, so $d = 2$. Each step fixes one red bit; no edge can fix two at once, so 2 steps is optimal.

Figure 5. Distance in Q_4 is the number of mismatched bits. The order of the flips does not matter: there are $h!$ shortest paths, here $2! = 2$.

Comprehensive example: distances from 0101 in Q_4

By the theorem, the vertices at distance j from 0101 are exactly the strings differing from it in exactly j positions, and there are $\binom{4}{j}$ of them (choose which j positions to flip):

$$1,\ 4,\ 6,\ 4,\ 1 \qquad (j = 0, 1, 2, 3, 4), \qquad \text{total } 16 = 2^4. \checkmark$$

At distance 2: 1001, 1111, 1100, 0011, 0000, 0110. And the *only* vertex at distance 4 is 1010, the bitwise complement of 0101.

Say both halves of the distance proof

A walk of length h only shows $d \leq h$. You also need $d \geq h$: "every edge changes exactly one bit, so it reduces the number of mismatches by at most one." Without this second sentence nothing rules out a shortcut.

Hypercubes in one breath

- Q_n: 2^n vertices, n-regular, bipartite; two copies of Q_{n-1} plus a matching.
- d-regular on v vertices $\Rightarrow e = dv/2$.
- $d(x, y)$ = number of differing bits; $\binom{n}{j}$ vertices at distance j from any vertex.

1.4 Gray codes: a path through every vertex of Q_n

The intuition

A **Gray code** lists all 2^n bit strings so that consecutive strings differ in one bit: a Hamilton path of Q_n. The **reflect-and-prefix** recipe builds one for n from one for $n-1$: write the old list with a 0 in front, then the old list *backwards* with a 1 in front. Inside each half consecutive strings still differ in one bit, and at the seam the two strings are the same old string with 0 and 1 in front.

Gray codes and Hamilton paths in Q_n

Theorem. For every $n \geq 1$, Q_n has a Hamilton path from 0^n to 10^{n-1}; since these two strings are adjacent, Q_n has a Hamilton cycle for every $n \geq 2$.
Proof by induction. $n = 1$: the path 0, 1. Step: let $P = x_1, \ldots, x_{2^{n-1}}$ be such a path in Q_{n-1}, from $x_1 = 0^{n-1}$ to $x_{2^{n-1}} = 10^{n-2}$. Then

$$0x_1,\ 0x_2,\ \ldots,\ 0x_{2^{n-1}},\ \ 1x_{2^{n-1}},\ \ldots,\ 1x_2,\ 1x_1$$

lists all 2^n strings once each; consecutive strings in each half differ in one bit (the bit that differed in P); the seam $0x_{2^{n-1}} \to 1x_{2^{n-1}}$ flips only the first bit. It starts at 0^n and ends at $1x_1 = 10^{n-1}$. ■

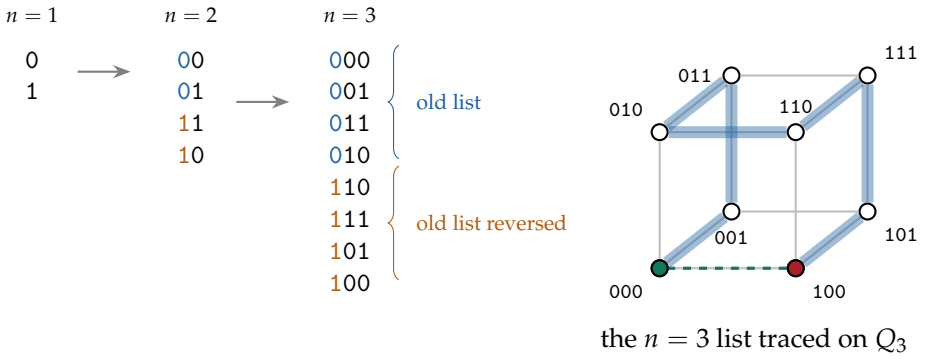


Figure 6. Reflect-and-prefix. The list for $n = 3$ is a Hamilton path from 000 (green) to 100 (red). The last and first strings also differ in one bit (dashed green), so it closes up into a Hamilton cycle.

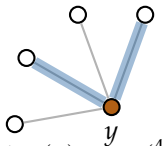
Gray codes in one breath

Reflect and prefix. Every Q_n ($n \geq 1$) has a Hamilton path, every Q_n ($n \geq 2$) a Hamilton cycle.

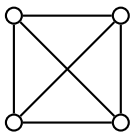
1.5 Counting paths: choose the middle first

The intuition

A path of length 2, $x - y - z$, is a "cherry": a middle vertex y with two of its neighbours. A path has **no direction**: x, y, z and z, y, x are the same path. Counting "start, then next, then next" counts every path twice. Counting by the **middle vertex** avoids that, because every path of length 2 has exactly one middle: pick y, then pick an unordered pair of neighbours.



$\deg(y) = 4$: $\binom{4}{2} = 6$ cherries centred at y



K_4

Paths of length 2 in K_4:
by middle: $4 \cdot \binom{3}{2} = 12$. ✓
"start, next, next": $4 \cdot 3 \cdot 2 = 24$, which counts each path **twice**.

Figure 7. Counting paths of length 2 by their middle vertex.

Paths of length 2

In any simple graph G, the number of paths of length 2 (as subgraphs, so x, y, z and z, y, x are the same path) is

$$\sum_{y \in V} \binom{\deg(y)}{2}.$$

Why: the map "path $\mapsto$ (middle vertex, set of its two ends)" is a bijection onto the pairs $(y, \{x, z\})$ with $x, z \in N(y)$, $x \neq z$; there are $\binom{\deg y}{2}$ choices of $\{x, z\}$ for each y. The equivalent ordered count is $\frac{1}{2} \sum_y \deg(y)\big(\deg(y) - 1\big)$.

Comprehensive example: the Petersen graph

The Petersen graph is 3-regular on 10 vertices, so it has $10 \cdot \binom{3}{2} = 30$ paths of length 2.
Paths of length 3, $w - x - y - z$. Count *ordered* sequences: 10 choices for w, 3 for x, 2 for y (any neighbour of x except w), 2 for z (any neighbour of y except x). Could $z = w$? Only if w, x, y were a triangle, and the Petersen graph has no triangle. So there are $10 \cdot 3 \cdot 2 \cdot 2 = 120$ ordered sequences, and each path is counted twice (once from each end): $\boxed{60}$ paths of length 3.

The factor of 2

Decide at the start whether you are counting *paths* (unordered) or *walks/sequences* (ordered), and say which in your answer. Every "start then step" count must be divided by 2 at the end, and you must check that the end vertices cannot coincide (that is what "no triangle" did above).

1.6 Building a graph with prescribed degrees

The intuition

"Prove that there *exists* a graph with ..." is a construction problem. The proof has two parts and both are compulsory: (1) **describe** the graph precisely (vertex set and an exact rule for the edges), (2) **verify** every required property (no loops, the degrees, connected, ...) from that rule. A picture for small cases helps the reader, but the rule is the proof.
Before constructing, check that the request is not impossible: the degree sum must be even, no degree can exceed $v - 1$, and a vertex of degree $v - 1$ is adjacent to *everybody*, which constrains all the others.

Simple example: an impossible request

Is there a simple graph with degrees $5, 5, 4, 3, 2, 1$? The sum is 20, even, so parity is fine. But $v = 6$, so each vertex of degree 5 is joined to all the other five vertices. Then the last vertex is joined to both of them and has degree ≥ 2, not 1. No such graph.

Necessary conditions and a standard reduction

If $d_1 \geq d_2 \geq \cdots \geq d_v$ are the degrees of a simple graph, then $\sum d_i$ is even and $d_1 \leq v - 1$.
Reduction (Havel–Hakimi). The list is realisable if and only if the list obtained by deleting d_1 and subtracting 1 from the next d_1 entries is realisable. (Join the top vertex to the d_1 vertices of largest remaining degree.) Repeating this either reaches all zeros (yes) or gets stuck, with a negative entry or a d_1 larger than the number of remaining entries (no), and running it backwards builds the graph.

Comprehensive example: realise $4, 3, 3, 2, 2, 2$

Sum 16, so $e = 8$.

$$4,3,3,2,2,2 \;\to\; 2,2,1,1,2 \;=\; 2,2,2,1,1 \;\to\; 1,1,1,1 \;\to\; 0,1,1 \;=\; 1,1,0 \;\to\; 0,0.$$

Reading it backwards with vertices $a, \ldots, f$: a is joined to b, c, d, e; then (degrees left b:2, c:2, f:2, d:1, e:1) b is joined to c, f; then c to f and d to e. Check the degrees from the edge list $ab, ac, ad, ae, bc, bf, cf, de$: a:4, b:3, c:3, d:2, e:2, f:2. ✓ Eight edges, connected. ✓

Constructions in one breath

Name the vertices, give an exact edge rule, then verify each property from the rule: loop-free, degrees, connectedness. Parity and "degree $v-1$ means adjacent to all" kill impossible requests fast.

1.7 The longest path trick

The intuition

Take a **longest** path $P = v_0, v_1, \ldots, v_m$ (it exists because the graph is finite). Its end v_0 cannot have a neighbour *off* the path, or we could stick that neighbour on the front and get a longer path. So **every neighbour of an end vertex lies on the path**. That single observation turns a lower bound on degrees into a lower bound on the length of P, or into a cycle.

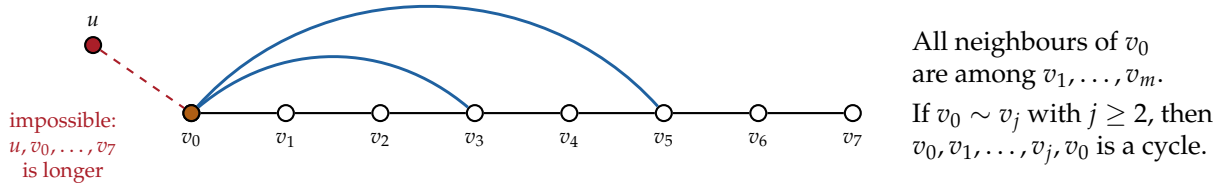


Figure 8. The end of a longest path has all its neighbours (blue) on the path. An outside neighbour (red) would extend it.

Simple example: a tree has a leaf

In a tree on ≥ 2 vertices take a longest path $v_0, \ldots, v_m$ ($m \geq 1$). All neighbours of v_0 lie on P; if $v_0 \sim v_j$ for some $j \geq 2$ we would get a cycle. So v_1 is its only neighbour: $\deg v_0 = 1$.

The lemma

Longest path lemma. Let G be a finite simple graph and $P = v_0, \ldots, v_m$ a path of maximum length in G. Then $N(v_0) \subseteq \{v_1, \ldots, v_m\}$ and $N(v_m) \subseteq \{v_0, \ldots, v_{m-1}\}$.

Comprehensive example: $\delta(G) \geq 2$ forces a cycle

Claim. If every vertex of a finite simple graph G has degree at least 2, then G contains a cycle.
Proof. Let $P = v_0, \ldots, v_m$ be a longest path. By the lemma all neighbours of v_0 lie in $\{v_1, \ldots, v_m\}$, and $\deg v_0 \geq 2$, so v_0 has a neighbour v_j other than v_1, with $j \geq 2$. Then $v_0, v_1, \ldots, v_j, v_0$ is a cycle (its length $j+1 \geq 3$, and its vertices are distinct). ■
Why "simple" matters. In a multigraph two vertices joined by two parallel edges have degree 2 each, and the only "cycle" would have length 2. The lemma uses "a neighbour other than v_1", which needs distinct neighbours, not just degree.

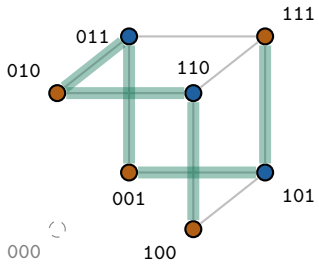
The greedy walk is not a proof by itself

"Start anywhere and keep walking to a new vertex" is the right idea, but written as "keep going until you are stuck" it hides the key point: *why* you can keep going and *what* you know when you are stuck. Phrase it with a longest (or maximal) path and the lemma, and the argument becomes three exact lines.

1.8 Paths in bipartite graphs

The intuition

In a bipartite graph every edge switches side, so a path is a sequence that **alternates** $X, Y, X, Y, \ldots$. Count how many vertices of each side such a sequence can contain: the two counts differ by at most 1. This is the same alternation that Tutorial 8 used to rule out Hamilton cycles.



Q_3 minus 000

- even weight: 3 vertices
- odd weight: 4 vertices

A path alternates colours, so it has at most 3 blue and $3 + 1 = 4$ orange vertices: at most 7 vertices, length at most 6.

Green: 100, 110, 010, 011, 001, 101, 111, length 6. Optimal.

A Hamilton *cycle* is impossible: it would need $3 = 4$.

Figure 9. Alternation in a bipartite graph. The path must start and end on the larger (orange) side to use all seven vertices.

Alternation

Alternation. Let G be bipartite with parts X, Y, and let P be a path in G. Then the numbers of vertices of P in X and in Y differ by at most 1.

Proof. Consecutive vertices of P lie in different parts, so P's vertices alternate between X and Y; an alternating sequence has equally many of each, or one more of the kind it starts (and ends) with. ■

Bipartite paths in one breath

- Paths alternate sides, so the numbers of vertices from each side differ by at most 1.
- The same alternation rules out Hamilton cycles when the sides have different sizes.

The toolbox at a glance

Situation	Tool
Are G, H isomorphic?	Yes: give the bijection. No: a differing invariant.
Edges or degrees of $\overline{G}$	$e + \bar{e} = \binom{n}{2}$, $\overline{\deg} = n - 1 - \deg$
Self-complementary?	bijection swapping edges and non-edges, checked on both kinds of pair
Edges of a regular graph	$e = dv/2$ (Handshaking)
Distance in Q_n	number of differing bits; $\binom{n}{j}$ vertices at distance j
Long paths in Q_n	Gray code (reflect and prefix): Hamilton path
Count short paths	choose the middle vertex; divide ordered counts by 2
"There exists a graph with …"	exact construction rule, then verify every property
Minimum degree ⇒ structure	longest path; its ends have all neighbours on it
Paths in bipartite graphs	alternation: side counts differ by ≤ 1

2 Mistakes and "spot the error"

Mistakes that cost marks

1. Concluding isomorphism from equal vertex count, edge count and degrees. Give the bijection.
2. Proving "not isomorphic" by failing to find a bijection. Name an invariant.
3. Calling G self-complementary because $e(G) = e(\overline{G})$.
4. Counting paths with an ordered "start, next, next" count and forgetting to divide by 2.
5. Claiming a longest path without proving that nothing longer exists (or the reverse: a bound with no path that attains it).
6. Stating $d(x, y) = h(x, y)$ in Q_n with only the "flip the bits" half.
7. In a construction, drawing one small picture and writing "and so on". Give the general rule and verify it.
8. In an induction on graphs, adding a vertex to a graph that already has the property *for* n (build-up error) instead of starting from an arbitrary graph *for* $n + 1$, when the statement is "for all graphs".

Spot the error 1

"*Proof that* $G \cong H$." Both graphs have 7 vertices, 9 edges and degree sequence $3, 3, 3, 3, 2, 2, 2$, so they are isomorphic.

Spot the error 2

"*Count of paths of length* 2 *in* K_5." Choose the first vertex in 5 ways, the second in 4, the third in 3: 60 paths.

Spot the error 3

"*Proof that* $\delta(G) \geq 2$ *gives a cycle, by induction on* $|V|$." True for 3 vertices (a triangle). Given G on n vertices with $\delta \geq 2$ and a cycle, add a vertex joined to two old vertices; the new graph on $n + 1$ vertices still has $\delta \geq 2$ and still has the old cycle.

Spot the error 4

"*Distance between* `00000` *and* `11100` *in* Q_5." The path `00000`, `10000`, `11000`, `11100` has length 3, so the distance is 3.

3 Practice problems

Try each one for ten minutes before you look at the appendix.

E1. Prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$. Then show that the complement of C_6 is not isomorphic to $K_{3,3}$, although both are 3-regular on 6 vertices.

E2. The complement of the Petersen graph P: how many edges does it have, and what are its degrees? Is $\overline{P}$ connected?

E3. In Q_5, how many vertices are at distance exactly 2 from `10110`? List the ones that agree with it in the first three positions.

E4. Prove that Q_n contains exactly $\binom{n}{2}2^{n-2}$ cycles of length 4 ($n \geq 2$). (Hint: a 4-cycle uses two coordinate directions.)
E5. Count the paths of length 3 in K_n.
E6. Is there a simple graph with degrees $4,4,3,3,2$? With degrees $3,3,3,1$? Construct it or prove there is none.
E7. Prove that a graph with $\delta(G) \geq 1$ in which every vertex has degree at most 2 is a disjoint union of paths and cycles. (Hint: look at a component and at a longest path in it.)
E8. The 3×3 grid graph (lattice points joined to horizontal and vertical neighbours) has 9 vertices. Show it has a Hamilton path but no Hamilton cycle. Does the 3×4 grid have a Hamilton cycle?
E9. Prove that Q_3 is isomorphic to $K_{4,4}$ with a perfect matching removed. (Hint: when is an even-weight string at distance 3 from an odd-weight one?)

A Answers to the "spot the error" puzzles

1. Equal invariants never prove isomorphism. One needs an explicit bijection (or the graphs may differ on a finer invariant, such as the number of triangles or whether the degree-2 vertices are adjacent).

2. $5 \cdot 4 \cdot 3 = 60$ counts *ordered* sequences; each path x, y, z is also counted as z, y, x. The answer is 30, which matches the middle-vertex count $5 \cdot \binom{4}{2} = 30$.

3. This is the build-up error. It only proves the claim for graphs that can be *grown* in that particular way, not for every graph on $n+1$ vertices with $\delta \geq 2$. (And the claim is easier anyway: use a longest path, Section 1.7.)

4. Only half the proof: the walk gives $d \leq 3$. Also needed: each edge changes one bit and the strings differ in 3 bits, so $d \geq 3$. (The answer 3 is right; the justification is incomplete.)

B Answers to the practice problems

E1. If f is an isomorphism $G \to H$ then $\{a,b\} \notin E(G) \iff \{f(a), f(b)\} \notin E(H)$, so the same f is an isomorphism $\overline{G} \to \overline{H}$; apply this to $\overline{G}$, $\overline{H}$ for the converse, since $\overline{\overline{G}} = G$. Now $\overline{C_6}$ is the prism (Section 1.2), which has triangles, while $K_{3,3}$ has none. (Equivalently, $\overline{K_{3,3}} = 2K_3$ is not $\cong C_6$.)

E2. $\binom{10}{2} - 15 = 30$ edges; every degree is $9 - 3 = 6$. Connected: take $x \neq y$. If they are non-adjacent in P they are adjacent in $\overline{P}$. If $x \sim y$ in P, pick any z adjacent to neither (there are $10 - 2 - 4 = 4$ such vertices, since x, y have 2 further neighbours each and none in common, as P has no triangle); then $x - z - y$ in $\overline{P}$.

E3. $\binom{5}{2} = 10$. Agreeing in the first three positions means flipping both of the last two: `10101` only.

E4. In a 4-cycle $x_0x_1x_2x_3$ the four edges flip coordinates i, j, i, j for some $i \neq j$ (to return to x_0 each coordinate must be flipped an even number of times, and consecutive flips of the same coordinate would repeat a vertex). So the cycle consists of the 4 strings obtained by fixing all coordinates except i, j. Choose $\{i, j\}$: $\binom{n}{2}$; fix the other $n-2$ bits: 2^{n-2}. Each choice gives exactly one 4-cycle.

E5. Ordered sequences of 4 distinct vertices: $n(n-1)(n-2)(n-3)$, all of them paths in K_n. Each path counted twice: $\frac{n(n-1)(n-2)(n-3)}{2}$. ($K_4$: 12.)

E6. $4,4,3,3,2$: sum 16, $v = 5$. The two degree-4 vertices are joined to everyone, which gives the degree-2 vertex its two edges; add the edge between the two degree-3 vertices. Edges: ab, ac, ad, ae,

bc, bd, be, cd, with degrees a:4, b:4, c:3, d:3, e:2. ✓ For $3,3,3,1$: $v = 4$, so each degree-3 vertex is joined to all others, and the last vertex then has degree 3, not 1. Impossible.

E7. Let C be a component; it has ≥ 2 vertices since $\delta \geq 1$. Take a longest path $P = v_0, \ldots, v_m$ in C ($m \geq 1$). Every interior vertex v_i already has its two allowed edges on P. By the lemma, $N(v_0) \subseteq \{v_1, \ldots, v_m\}$, and v_0 cannot be adjacent to an interior v_j with $j \geq 2$ (that vertex is full), so $N(v_0) \subseteq \{v_1, v_m\}$; similarly for v_m. *If* $v_0 \sim v_m$ (possible only if $m \geq 2$), then $v_0, \ldots, v_m, v_0$ is a cycle using two edges at each of its vertices. *If not*, v_0 and v_m have degree 1. In both cases no vertex of P has an edge leaving $V(P)$, and C is connected, so C has no other vertex: C is a cycle or a path.

E8. Colour by the parity of $i + j$: 5 even, 4 odd. A Hamilton cycle would alternate and need $5 = 4$: none. A snake (row 1 left to right, row 2 right to left, row 3 left to right) is a Hamilton path. The 3×4 grid has $6 + 6$, so there is no obstruction, and indeed (rows 1..3, columns 1..4) $(1,1), (1,2), (1,3), (1,4), (2,4), (3,4), (3,3), (2,3), (2,2), (3,2), (3,1), (2,1), (1,1)$ is a Hamilton cycle.

E9. Let A be the 4 even-weight strings and B the 4 odd-weight strings of length 3. Every edge of Q_3 joins A to B. For $x \in A$, $y \in B$ the Hamming distance $h(x, y)$ is odd (weights of different parity), so $h(x, y) \in \{1, 3\}$, and $h(x, y) = 3$ exactly when $y = \bar{x}$, the bitwise complement of x. So x is adjacent to every vertex of B except $\bar{x}$. Hence Q_3 is $K_{4,4}$ (parts A, B) minus the perfect matching $\{x, \bar{x}\}$, $x \in A$: the identity map is the isomorphism.